

Cyclotron Resonance Spectroscopy in a High Mobility Two-Dimensional Electron Gas Using Characteristic Matrix Methods

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Reference: [Hilton, 2012]

Schema

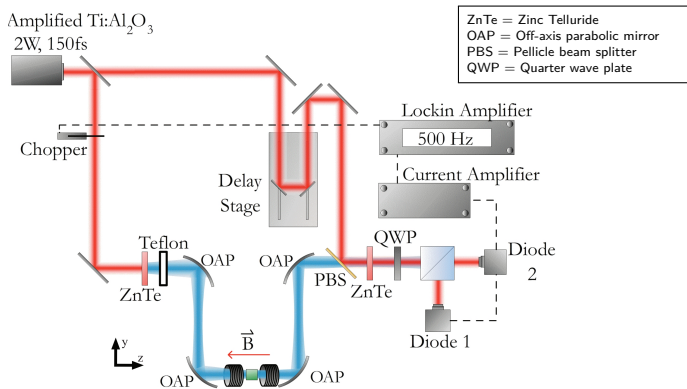


Figure: A diagram of a typical terahertz time-domain spectrometer with an external magnetic field in the Faraday geometry ($\vec{B} \perp 2DEG$). There exist four quartz windows on either the entrance or exit of the split coil magnet, which generate satellite pulses in addition to the gallium arsenide 2DEG/Substrate. The $\hat{y} - \hat{z}$ 2DEG sample coordinate system, as defined in the manuscript, is indicated in this diagram. The polarization of the input terahertz pulse is perpendicular to the page (\hat{x}), while the transmitted pulse field is polarized in the $\hat{x} - \hat{y}$ plane.

Basic Process

- **Pulse generation:** Electric field of the 100fs pulse as a Gaussian-modulated cosine is generated by Ti:Sapphire laser.
- **Optical rectification:** THz polarization is parallel to the incoming laser beam's polarization, generating a THz pulse due to second order non-linearity by optical rectification in a ZnTe crystal, *linearly polarized say along \hat{x}* .
- Under proper phase matching, results in the emission of a coherent broadband spectrum with a bandwidth of approximately $\Delta\nu = 0.8 THz = \frac{1}{2\pi\sigma} = \frac{c}{2\pi d}(n_c - 1)$ that is centered near $\nu_0 = 0.5 THz$ and has $|E_0| \sim 10 kV/m$.
- Can be modeled using a Gaussian-modulated sine pulse:

$$E(t) = E_0 \cdot \exp\left(-\frac{(t - t_0)^2}{2\sigma^2}\right) \cdot \sin(2\pi\nu_0(t - t_0) + \pi)$$

Basic Process

Induced by Landau quantization in 2DEG:

- **Circular dichroism** causes differential absorption or rotation of the left- and right-circularly polarized components of the light, which means that the polarization is affected by the material's magnetic-field-dependent properties (breaking of TRS).
- **Birefringence** (different refractive indices) causes a phase difference between the orthogonal polarization components of the pulse.
- hence, the transmitted pulse is elliptically polarized.

Basic Process

- **Linear electrooptic effect:** A THz pulse changes ZnTe crystal's refractive index: $\Delta n(t) = n_c \cdot r_{41} \cdot E(t)$, where r_{41} is the non-zero electro-optic tensor.
- ZnTe crystals are cut along the (110) plane.
- Effective Pockel's response is maximized when THz is polarized along [001].
- Probe beam is at 45° , to the principal axes of the (110) plane.
- For our setup, [001] is aligned along \hat{y} for maximum Pockel's response.
- We measure this component for both $\pm B_{ext}$ to ensure proper alignment (100:1 polarization extinction ratio).

THz Time Domain Waveforms

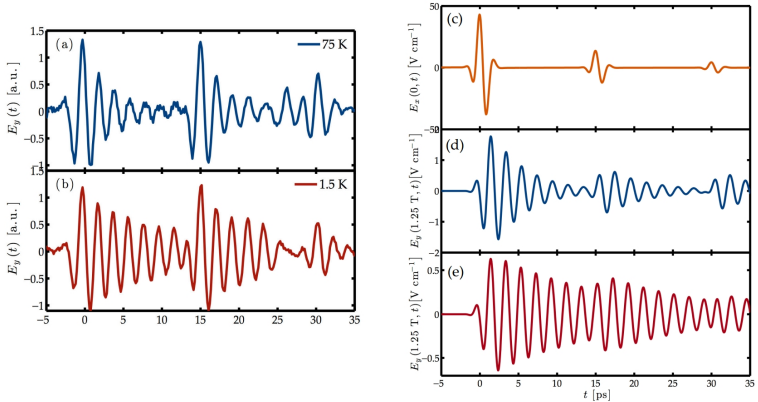


Figure: Terahertz time-domain waveforms taken in the high mobility 2DEG, $E(t)$ for different Decay lifetimes T_2 in $B_{ext} = 1.25 T$. a and b are experimental data. c, d and e are simulations for E_x , E_y at $T_2 = 0.5 ps$ and E_y at $T_2 = 15.1 ps$ respectively. (best fit for a and b).

Coherence in 2DEG

- B_{ext} is aligned in z-direction, perpendicular to 2DEG, causing continuous energy spectrum of free electrons quantizes into discrete energy levels known as Landau levels (LLs).

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

- Energy spacing, $\Delta E = \hbar eB/m^*$ a cyclotron frequency $\omega_c = \frac{eB_{\text{ext}}}{m^*}$.
- THz is tuned to resonance with Landau energy spacing between highest filled $|n\rangle$ and $|n+1\rangle$ levels.
- Results in formation of coherent superposition of states.

Coherence in 2DEG

- **Superposition of states:** A and B are complex coefficients that evolve over time with well defined relative phase of ω_c (coherence).

$$|\psi(t)\rangle = Ae^{-iE_n t/\hbar} |n\rangle + Be^{-iE_{n+1} t/\hbar} |n+1\rangle$$

- **Dephasing:** Due to Scattering events, Inhomogeneity of B_{ext} , and finite temperature, coherence is lost over time. Decay lifetime T_2 such that:

$$E_y(t) \propto e^{-t/T_2} \cos(\omega_c t + \phi)$$

- Hence need Low temperature and high mobility.

Satellite Pulses

- finite thickness of the eight quartz magnet windows and gallium arsenide substrate lead to formation of a series of satellite terahertz pulses
- Delays determined by their thicknesses and the refractive index of quartz or GaAs.
- Etalon effect (Fabry-Pérot) due to thin-film interference. We take a substrate with thickness 7.5 times wavelength of 1 thz.
- Small $\Delta t \approx 15ps$ between the two pulses allows satellite pulse to overlap with tail of previous oscillating field, leading to interference.

Solving for Satellite Pulses

- **Introducing wedge:** primary and satellite pulses travel with different propagation vector directions, allowing separation.
- **Windowing:** Crop time domain data behind satellite pulse before using fft. Limited by frequency resolution of acquired data.
- Need TMM would allow analysis of long cyclotron decoherence lifetimes
- Future application: Use of time-delayed satellite pulses is an experimental method of coherent control of the quantum superposition states.

Defining basis

Due to elliptical polarization, in $x - y$ plane of our original basis:

$$[\hat{x}, \hat{y}, \hat{z}]$$

We define a new basis:

$$[\hat{\sigma}_+, \hat{\sigma}_-, \hat{z}]$$

with,

$$\hat{\sigma}_{\pm} = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$$

And monochromatic components of the total electromagnetic field, polarized in the $x - y$ plane, can be written:

$$\begin{aligned}\vec{E}(z, \nu) &= U(z, \nu)\hat{\sigma}_+ + P(z, \nu)\hat{\sigma}_- \\ \vec{H}(z, \nu) &= V(z, \nu)\hat{\sigma}_+ - Q(z, \nu)\hat{\sigma}_-\end{aligned}\tag{1}$$

Anisotropic Materials

- **Anisotropic materials:** exhibit different properties in different directions like Birefringence and Circular dichroism.
- **Dielectric tensor:** $\bar{\epsilon}$ describes anisotropic materials with broken TRS but no birefringence.

$$\bar{\epsilon}(\mathbf{z}, \nu) = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ -\epsilon_{xy} & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix} \equiv \begin{bmatrix} \epsilon_{pp} & 0 & 0 \\ 0 & \epsilon_{mm} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \quad (2)$$

- $\epsilon_{pp} = \epsilon_{xx} + i\epsilon_{xy}$ and $\epsilon_{mm} = \epsilon_{yy} - i\epsilon_{xy}$.
- $\epsilon_{pp} = \epsilon_{mm}$ for isotropic materials with TRS, since, $\epsilon_{xy} = 0$.
- $\bar{\mu}$ is diagonal matrix with all elements μ .

Maxwell equations

Faraday's and Ampere's laws in the frequency domain:

$$\begin{aligned}\nabla \times E(z, \nu) &= +i2\pi\nu\mu(z, \nu)H(z, \nu) \\ \nabla \times H(z, \nu) &= -i2\pi\nu\epsilon(z, \nu)E(z, \nu)\end{aligned}\tag{3}$$

This when applied to (1) gives:

$$\begin{aligned}\frac{dU}{dz} &= -2\pi\nu\mu V \\ \frac{dV}{dz} &= +2\pi\nu\epsilon U \\ \frac{dP}{dz} &= +2\pi\nu\mu Q \\ \frac{dQ}{dz} &= -2\pi\nu\epsilon P\end{aligned}\tag{4}$$

This shows U and V ($\hat{\sigma}_+$ component) propagate independent of P and Q ($\hat{\sigma}_-$ component).

Propagation

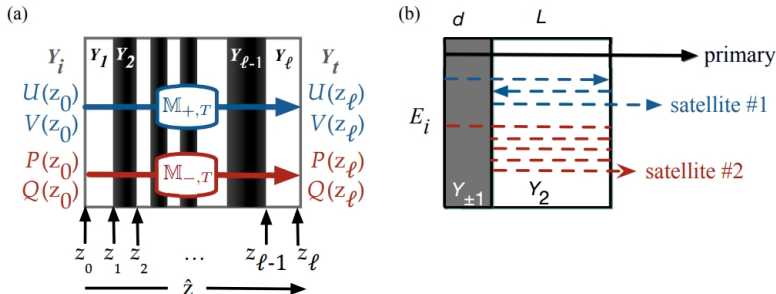


Figure: Propagation of a normally incident electromagnetic wave through an arbitrary stratified medium with l layers located at $z \in [z_0, z_1 \cdots z_{l-1}, z_l]$ with Admittance Y_j . 2DEG is Y_1 and GaAs substrate is Y_2 .

Propagating Waves

For a layer $z_j \leq z \leq z_{j+1}$, the solution to the (4) is:

$$\begin{aligned} P(z) &= P(z_j) \cos [\kappa_-(z - z_j)] + Q(z_j) \left\{ +Y_-^{-1} \sin [\kappa_-(z - z_j)] \right\} \\ Q(z) &= Q(z_j) \cos [\kappa_-(z - z_j)] + P(z_j) \left\{ -Y_-^{-1} \sin [\kappa_-(z - z_j)] \right\} \\ U(z) &= U(z_j) \cos [\kappa_+(z - z_j)] + V(z_j) \left\{ -Y_+^{-1} \sin [\kappa_+(z - z_j)] \right\} \\ V(z) &= V(z_j) \cos [\kappa_+(z - z_j)] + U(z_j) \left\{ +Y_+ \sin [\kappa_+(z - z_j)] \right\} \end{aligned} \quad (5)$$

Where Admittance is given by $Y_+ = \sqrt{\varepsilon_{pp}\mu^{-1}}$, $Y_- = \sqrt{\varepsilon_{mm}\mu^{-1}}$ and complex propagation vector magnitudes $\kappa_+ = 2\pi\nu\sqrt{\varepsilon_{pp}\mu}$, $\kappa_- = 2\pi\nu\sqrt{\varepsilon_{mm}\mu}$.

Characteristic Matrix Method

Lets write components of the electromagnetic wave in $\hat{\sigma}_{\pm}$:

$$Q_{+}(z, \nu) = \begin{bmatrix} U(z, \nu) \\ V(z, \nu) \end{bmatrix} \quad Q_{-}(z, \nu) = \begin{bmatrix} P(z, \nu) \\ Q(z, \nu) \end{bmatrix} \quad (6)$$

If both Q_{\pm} are known at z_j , the Q_{\pm} at $z_{j+1} = z_j + d$ can be found using,

$$Q_{\pm}(z_j, \nu) = \bar{M}_{\pm,j} Q_{\pm}(z_{j+1}, \nu) \quad (7)$$

with propagation is described by

$$\bar{M}_{\pm,j} = \begin{bmatrix} \cos(\kappa_{\pm} d) & -Y_{\pm}^{-1} \sin(\kappa_{\pm} d) \\ -Y_{\pm} \sin(\kappa_{\pm} d) & \cos(\kappa_{\pm} d) \end{bmatrix} \quad (8)$$

Total Transfer matrix

We could define total transfer matrix $\bar{M}_{\pm,T}$ such that it is product of all individual transfer matrices:

$$\begin{aligned}\bar{M}_{\pm,T} &= \bar{M}_{\pm,1} \bar{M}_{\pm,2} \cdots \bar{M}_{\pm,l} \\ &= \begin{bmatrix} A_{\pm} & B_{\pm} \\ C_{\pm} & D_{\pm} \end{bmatrix}\end{aligned}$$

Where at $z < z_0$ before hitting interface is described by (1) with superposition of incident and reflected field.

Total Transfer matrix

We also get following relations for our two layer approximation as:

$$\begin{aligned}A_{\pm} &= \sin \varphi_{\pm} \sin \theta - Y_{\pm,1}^{-1} Y_2 \cos \varphi_{\pm} \cos \theta \\B_{\pm} &= \pm \left[Y_2^{-1} \cos \theta \sin \varphi_{\pm} + Y_{\pm,1}^{-1} \cos \varphi_{\pm} \sin \theta \right] \\C_{\pm} &= \mp \left[Y_2 \cos \theta \sin \varphi_{\pm} + Y_{\pm,1} \cos \varphi_{\pm} \sin \theta \right] \\D_{\pm} &= \sin \varphi_{\pm} \sin \theta - Y_{\pm,1} Y_2^{-1} \cos \varphi_{\pm} \cos \theta\end{aligned}\tag{9}$$

Where $\varphi_{\pm} = \kappa_{\pm,1}d$ and $\theta = \kappa_2 L$. Remember, $Y_+ = \sqrt{\epsilon_{pp}\mu^{-1}}$, $Y_- = \sqrt{\epsilon_{mm}\mu^{-1}}$ and $\kappa_+ = 2\pi\nu\sqrt{\epsilon_{pp}\mu}$, $\kappa_- = 2\pi\nu\sqrt{\epsilon_{mm}\mu}$.

Finding Permittivity

For cyclotron resonance-active polarization $\hat{\sigma}_+$, the susceptibility is

$$\bar{\chi}(\nu) = \chi_0 \frac{i - 2\pi(\nu - \nu_{CR})T_2}{1 + 4\pi^2(\nu - \nu_{CR})^2 T_2^2},$$

with imaginary and real parts describing circular dichroism and birefringence. The permittivity is

$$\varepsilon_{pp} = \varepsilon_b + (1 + \bar{\chi}(\nu))\varepsilon_0.$$

Selection rules enforce $\Delta m = 1$, allowing $\hat{\sigma}_+$ ($m = 1$) to couple $|n\rangle \rightarrow |n+1\rangle$. $\hat{\sigma}_-$ ($m = -1$) cannot, so

$$\varepsilon_{mm} = \varepsilon_b.$$

Transmission Matrix

$$t_+ \equiv \frac{U_t(z_\ell)}{U_i(z_0)} \quad (10a)$$

$$t_- \equiv \frac{P_t(z_\ell)}{P_i(z_0)} \quad (10b)$$

Boundary Conditions:

- At input: $E(z_0) = E_i + E_r$, $H(z_0) = Y_i(E_i - E_r)$.
- At output: $E(z_\ell) = E_t$, $H(z_\ell) = Y_t E_t$.

Using matrix propagation, the field relations yield:

$$\begin{bmatrix} E_i + E_r \\ Y_i(E_i - E_r) \end{bmatrix} = \overline{\mathbb{M}}_{\pm, T} \begin{bmatrix} E_t \\ Y_t E_t \end{bmatrix}.$$

Transmission Matrix

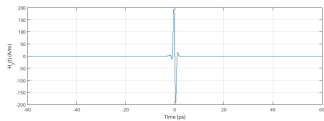
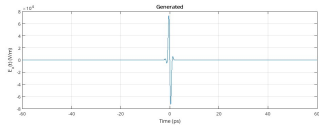
Solving for E_r and t_{\pm} , the two linear equations give:

$$t_{\pm} = \frac{2Y_i}{(C_{\pm} + D_{\pm}Y_t) + Y_i(A_{\pm} + B_{\pm}Y_t)}.$$

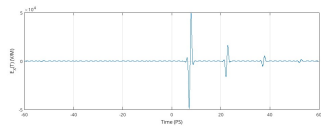
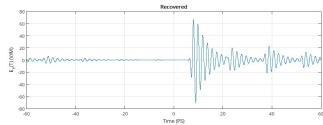
Factor out Y_i^{-1} , giving the field transmission coefficient for the $\hat{\sigma}_{\pm}$ component is:

$$t_{\pm}(\nu) = \frac{2}{(A_{\pm} + Y_t Y_i^{-1} D_{\pm}) \pm i(Y_i^{-1} C_{\pm} - Y_t B_{\pm})} \quad (11)$$

Our Simulation Result



(a) Generated



(b) Recovered

Figure: Generated and Recovered $E(t)$.

**Transfer matrix method for precise
determination of thicknesses in a 150-ply
polyethylene composite material**

Reference: [Palka et al., 2015]

Material

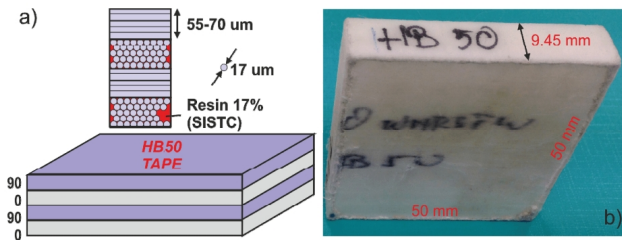


Figure: 154 plies of average thickness $61 \pm 7 \mu\text{m}$ refractive index $n_c = 1.521 + i0.002$, i.e birefringence $\Delta n = 0.04$.

Result

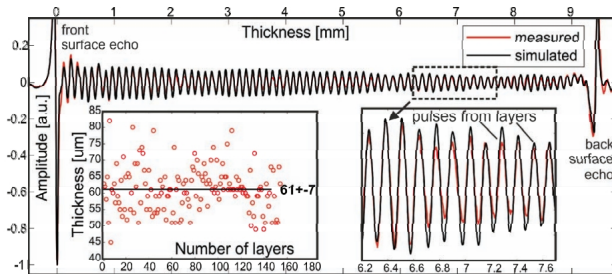


Figure: Waveform reflected from the sample - measured and simulated

The reflection of a terahertz (THz) pulse from a multilayer system is described using transfer matrices. The interface matrix between the i -th and $(i + 1)$ -th layers:

$$D_{i,j} = \frac{1}{t_{ij}} \begin{bmatrix} 1 & r_{ij} \\ r_{ij} & 1 \end{bmatrix} \quad (10)$$

where r_{ij} and t_{ij} are the Fresnel coefficients.

The propagation matrix for the i -th layer:

$$P_i(\omega) = \begin{bmatrix} \exp\left(\frac{i\omega N_i d_i}{c}\right) & 0 \\ 0 & \exp\left(-\frac{i\omega N_i d_i}{c}\right) \end{bmatrix} \quad (11)$$

where ω is the angular frequency, N_i is the refractive index, d_i is the thickness, and c is the speed of light.

The total transfer matrix:

$$M_{\text{total}}(\omega) = \prod_{i=0}^k P_i(\omega) \cdot D_{i,i+1} = \begin{bmatrix} M_{11}(\omega) & M_{12}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) \end{bmatrix} \quad (12)$$

The reflection coefficient:

$$R(\omega) = \frac{M_{21}(\omega)}{M_{11}(\omega)} \quad (13)$$

The reflected THz signal:



$$E_r(t) = F^{-1} [R(\omega) \cdot F[E_0(t)]] \quad (14)$$

where F denotes the Fourier transform.

The time-domain fitting error function:

$$QERR(d_1, \dots, d_k) = \sum_t |\text{sign}_{\text{meas}}(t) - \text{sign}_{\text{sim}}(d_1, \dots, d_k, t)| \quad (15)$$

Discussed References

-  Hilton, D. J. (2012).
Cyclotron resonance spectroscopy in a high mobility two dimensional electron gas using characteristic matrix methods.
Opt. Express, 20(28):29717–29726.
-  Palka, N., Krimi, S., Ospald, F., Beigang, R., and Miedzinska, D. (2015).
Transfer matrix method for precise determination of thicknesses in a 150-ply polyethylene composite material.
In 2015 40th International Conference on Infrared, Millimeter, and Terahertz waves (IRMMW-THz), pages 1–2.