Landauer-Büttiker vs. Kubo Formalisms in Nanoscale Transport

A Comparative Study in 1D Interacting Systems

Term Project for the Course
Condensed Matter Physics II (PH5103)

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Abstract

This report investigates the conflict between Landauer-Büttiker scattering formalism and the Kubo linear response formalism in quantum wires. While standard Fermi liquid theory breaks down in one dimension quantum wires following the Tomonaga-Luttinger liquid model, experimental conductance remains quantized at $2e^2/h$. We reconcile these formalisms by analyzing the role of reservoirs and boundary conditions. Utilizing bosonization techniques, we derive the exact conductance for an interacting wire connected to non-interacting leads, demonstrating that the "washing out" of interaction parameters by the leads restores universality in DC transport, while the intrinsic conductivity of an isolated wire, as described by Kubo formalism, remains renormalized.

Acknowledgement

I would like to express my sincere gratitude to my course instructor, **Prof. Amit Ghosal**, for his invaluable guidance and for providing the conceptual framework necessary to undertake this study. His insightful lectures on advanced condensed matter physics were instrumental in shaping the structure and content of this report. The course was thoughtfully designed to emphasize physical concepts and their application to real systems, rather than overwhelming students with excessive formalism. Prof. Ghosal's consistent support and thoughtful insights throughout the term project were greatly appreciated and played a key role in our learning process.

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Chapter 1

Introduction to Problem Statements

1.1 Overview

The study of electron transport at the nanoscale has revealed fundamental quantization phenomena that challenge our classical understanding of resistance. In macroscopic conductors, transport is diffusive, governed by Ohm's law and material-specific conductivity (σ). However, as device dimensions shrink below the electron mean free path and phase coherence length, transport becomes ballistic. In this regime, the conductance (g) of a one-dimensional (1D) channel is quantized in units of the conductance quantum, $g_0 = 2e^2/h$. This Term paper primarily revolves around the solution to this problem as discussed by Safi [4] [5].

This quantization is successfully described by the **Landauer-Büttiker formalism**, which treats transport as a quantum mechanical scattering problem. In this view, current is determined by the transmission probability (T) of electron waves travelling between reservoirs. For a perfect ballistic wire connected to ideal reservoirs, $T = 1$, yielding the universal conductance $2e^2/h$.

However, a fundamental theoretical difficulty arises when electron-electron interactions are introduced. In 2D and 3D systems, interactions are screened, and Landau's Fermi Liquid theory ensures that quasiparticles effectively behave as non-interacting electrons as presented by Brooks [1] and Nikolic [2]. In 1D, this picture collapses. Electrons cannot pass one another, and any perturbation generates collective density excitations rather than single-particle excitations. The resulting state is a **Tomonaga-Luttinger Liquid (TLL)**.

Theoretical calculations for an infinite, isolated Luttinger liquid using the **Kubo formalism** predict a renormalized conductance:

$$g = K\frac{2e^2}{h} \tag{1.1}$$

where K is the Luttinger interaction parameter ($K < 1$ for repulsive interactions). This contradicts the Landauer result and experimental observations in high-mobility quantum wires, which typically show conductance plateaus at $2e^2/h$, not renormalized values.

This report addresses this dichotomy:

1. **The Conflict:** Why does the Kubo formalism applied to an isolated wire predict interaction-dependent conductance, while Landauer-Büttiker predicts universality?
2. **The Resolution:** How do the physical boundary conditions imposed by non-interacting 3D leads ("reservoirs") affect the measurement?

Formula	Formalism	Valid (Isolated)?	Valid (Leads)?
$g = \frac{2e^2}{h}\sum_n T_n$	**Landauer-Büttiker** Conductance is *transmission*	No Needs reservoirs	**Yes** Designed for this
$g' = \frac{2e^2}{h}K'$	**Kubo (Intrinsic)** Conductance is *local response*	**Yes** Gives $g' \neq g$	No Ignores boundaries
$g = \frac{2e^2}{h}T$	**Kubo (Open System)** Full response of wire+leads	No Needs reservoirs	**Yes** Proven Computationally equivalent to LB via DMRG.

Table 1.1: Comparison of Landauer and Kubo formalisms for the two physical scenarios, which are not equal only for non-interacting case. DMRG = density matrix renormalization group.

1.2 The Physical System

We analyze transport through a 1D quantum wire, which acts as the sample or scattering region, connected to two macroscopic leads that are in turn connected to reservoirs. This problem becomes important for nanoscale transport (nearly perfect 1D wires like

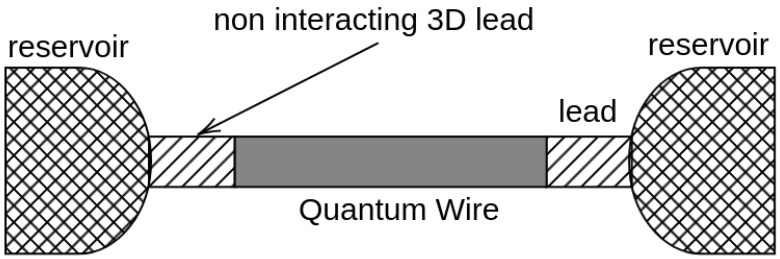

Figure 1.1: 3d leads connected to 1D interacting wire

Carbon Nanotubes or Semiconductor Nanowires) where conductance is quantized, set by the number of open channels at E_f (fermi energy).

1.3 How We Will Proceed

We will proceed by first establishing the formalism of transport in non-interacting systems (Chapter 2). We will then derive the bosonization technique to handle 1D interactions (Chapter 3) and construct the Hamiltonian for the interacting wire (Chapter 4). Finally, we will rigorously derive the conductance in the presence of leads to demonstrate the "washing out" of interaction effects (Chapter 5) and compare this with the intrinsic response of the wire (Chapter 6). Basically, the systems compare as follows:

Case 1: Realistic (Wire + Leads)

- **System:** The interacting 1D wire is connected to non-interacting 3D leads.
- **Question:** What is the two-terminal conductance $g = I/V$?
- **Method:** Landauer-Büttiker formalism, which is defined by transmission T between reservoirs.

- **Answer:** $g = \frac{2e^2}{h}$

Case 2: Theoretical (Isolated Wire)

- **System:** An infinite, isolated 1D interacting wire.
- **Question:** What is the "intrinsic" conductivity σ of the material?
- **Method:** Kubo formalism, defined as a response to a local electric field E_{loc}.
- **Answer:** $g' \neq \frac{2e^2}{h}$

While we did not cover the proves in the presentation due to time constraints, we try to cover all the derivations more regorously here.

1.4 Experimental Reality

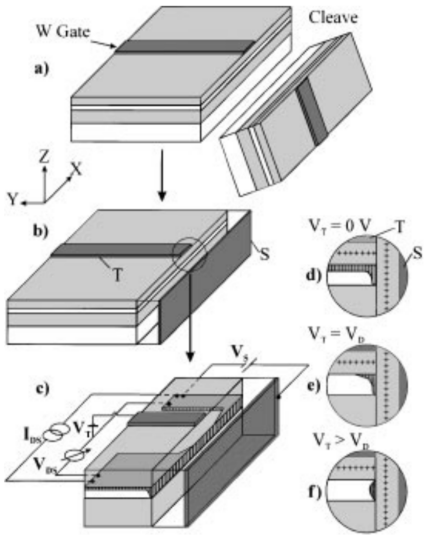

Figure 1.2: Figure 1 in the paper illustrates the intricate fabrication of a high-quality quantum wire using the "cleaved edge overgrowth" technique. (source: A. Yacoby et. al. (1996) [7]).

Let us look at the figure above and try to understand what is gate voltage. Further understanding on number on channels in 3D Landau formalism are discussed in subsection 2.3.2. The process starts with a modulation-doped GaAs/AlGaAs quantum well containing a 2D electron gas (2DEG). A tungsten top gate is patterned on the surface. The sample is then cleaved in ultra-high vacuum, exposing an atomically smooth edge which is immediately overgrown with a second modulation-doping layer. This creates a strong attractive potential at the edge, forming 1D edge states. To isolate a specific 1D wire segment, a negative voltage (V_T) is applied to the top gate, depleting the 2DEG beneath it while leaving the 1D edge channel intact. A side gate (V_S) then tunes the electron density in this wire, enabling precise control of conductance.

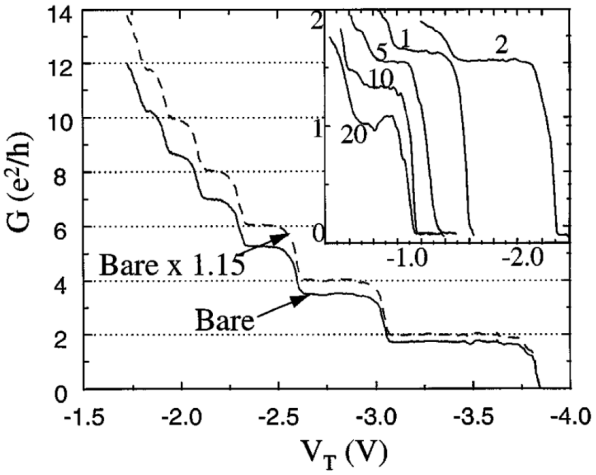

Figure 1.3: Conductance vs. Gate Voltage. Conductance is quantized but reduced by $\sim$ 25% ($0.75 \times 2e^2/h$). Plateaus are flat (independent of density), ruling out simple disorder models. (source: A. Yacoby et. al. (1996) [7])

Safi's theory predicts $g = 2e^2/h$ for a clean wire connected to leads. However, experiments on ultra-clean quantum wires show a persistent deviation.

Given in the above figure we see that A. Yacoby et. al. (1996)[7], showed that the conductance of a quantum wire deviates from the predicted result by around 25%. The small inset diagram above shows the last plateau for different lengths (in microns) of wires fabricated consecutively along the cleaved edge overgrowth. They show deviation from ideal behaviour as the length increeases due to increasing inpurities introduced in wire and also due to suspected lack of accounting of long range interactions [4] which were included for by Safi 1 year after the experiment [5].

- Yacoby argues the leads themselves (edge states) might be interacting (Non-Fermi Liquid).
- The coupling between the 2D reservoir and 1D wire is not "perfect transmission". Backscattering at the contact (due to momentum mismatch) likely causes the reduction.

Recent experiments by Rech et al. (PRL, 2009) [3] proved that for a fully equilibriated quantum wire,

$$g = \frac{2e^2}{h}\left[1 - \frac{\pi^2}{12}\left(\frac{T}{\mu}\right)^2\right]$$

with T giving temperature in kelvin and μ is chemical potential. This result strenthens the arguement made by Safi in her 1995 and 1997 papers and confirm that with rightly done sotup, such deviation must noot be expected. Safi however did not account for temperature scaling (as in power law), which is found in this result given, as the Luttinger-liquid theory used there does not account for the relaxation process. Small temperature-dependent corrections to quantized conductance have been observed in quantum point contacts. The latter are essentially short quantum wires, with only a few electrons in the one-dimensional part of the device.

Chapter 2

Overview of Transport Systems

Before addressing the complexities of the Luttinger liquid, it is essential to establish the quantum mechanical description of current and the derivation of conductance in non-interacting systems. This chapter reviews the fundamental concepts of probability currents and extends the analysis from 1D to 3D systems, forming the baseline for the Landauer formula.

2.1 Quantum Currents in One Dimension

2.1.1 Probability Currents

In quantum mechanics, the flow of particles is described by the probability current density $J(x,t)$. The conservation of probability is expressed by the continuity equation:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0 \tag{2.1}$$

where $\rho(x,t) = |\Psi(x,t)|^2$ is the probability density. For a particle of mass m described by a wavefunction $\Psi(x,t)$, the probability current is defined as:

$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^*}{\partial x} - \Psi^*\frac{\partial \Psi}{\partial x}\right) \tag{2.2}$$

For a stationary state $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$, the probability density is time-independent ($\partial\rho/\partial t = 0$), implying that the current J is spatially uniform ($\partial J/\partial x = 0$).

Consider a free particle represented by a plane wave $\psi(x) = Ae^{ikx}$. The particle density is $|A|^2$, and the current is:

$$J = \frac{\hbar k}{m}|A|^2 = v\rho \tag{2.3}$$

where $v = \hbar k/m$ is the velocity. The electrical current I carried by charge $q = -e$ is simply $I = -eJ$.

2.1.2 Scattering and Transmission

In a typical transport setup, a scattering region (the sample) is connected to two leads. We model the leads as regions of constant potential. An electron incident from the left lead ($x < 0$) with wavevector k is described by:

$$\psi_L(x) = Ae^{ikx} + Be^{-ikx} \tag{2.4}$$

In the right lead ($x > L$), the transmitted wave is:

$$\psi_R(x) = Fe^{ikx} \tag{2.5}$$

The transmission coefficient T is defined as the ratio of transmitted to incident probability flux:

$$T = \frac{J_{trans}}{J_{inc}} = \frac{|F|^2}{|A|^2} \tag{2.6}$$

Current conservation implies $R + T = 1$, where R is the reflection coefficient.

2.2 Landauer Formula in 1D interacting Case

The Landauer formula relates the macroscopic conductance to the microscopic transmission probability. We consider a 1D wire connected to two large reservoirs (contacts) with chemical potentials μ_L and μ_R.

2.2.1 Density of States and Current

We assume the reservoirs inject electrons into the wire up to their respective chemical potentials. To derive the current rigorously, we first consider the 1D wire of length L with quantized wave functions. The energy levels and wave functions are given by:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}; \quad \psi_n = \frac{1}{i\sqrt{2L}}(e^{ik_n x} + e^{-ik_n x}) \tag{2.7}$$

The wave vectors are quantized as $k_n = n\pi/L$, and the spacing between states in k-space is $\Delta k = \pi/L$. For a single normalized traveling wave, the electron density is $\rho = 1/2L$.

An electron in state k moving with velocity v_k carries a contribution to the current given by:

$$I_{state} = -e \cdot v_k \cdot \rho = -\frac{ev_k}{2L} \tag{2.8}$$

To find the total current I_{in} injected by the reservoir, we sum the contributions of all states k accounting for spin degeneracy (factor of 2):

$$I_{in} = 2\sum_k I_{state} = 2\sum_k \left(-\frac{ev_k}{2L}\right) \tag{2.9}$$

Since L is macroscopic, we convert this discrete sum to an integral using the density of states in k-space ($\sum \to \frac{L}{\pi}\int dk$):

$$I_{in} = -\frac{e}{L}\left(\frac{L}{\pi}\int v(k)dk\right) = -\frac{e}{\pi}\int v(k)dk \tag{2.10}$$

We now relate the particle velocity to the group velocity of the wave packet:

$$v(k) = \frac{\hbar k}{m} = \frac{1}{\hbar}\frac{dE}{dk} \tag{2.11}$$

Substituting this into our current integral allows us to change the integration variable from momentum k to Energy E:

$$\int v(k)dk = \int \left(\frac{1}{\hbar}\frac{dE}{dk}\right)dk = \frac{1}{\hbar}\int dE \tag{2.12}$$

This demonstrates that the dk terms cancel perfectly. Integrating over the bias window $\Delta V = -eU$ yields the incident current:

$$I_{in} = -\frac{e}{\pi\hbar}(\Delta V) = \frac{e^2}{\pi\hbar}U \tag{2.13}$$

The total transmitted current is then $I_{total} = I_{in}T$.

Proof instead, as described by Brooks

Equivalently, this result can be derived by considering the density of states per unit energy, dn/dE. The net current is the difference between the current flowing left-to-right and right-to-left within the energy window $\mu_L - \mu_R = -eV$:

$$I = \int_{\mu_R}^{\mu_L} (-e)v(k)T(E)\frac{dn}{dE}dE \tag{2.14}$$

Here, dn/dE is the density of states per unit energy. A crucial cancellation occurs in 1D. The density of states is:

$$\frac{dn}{dE} = 2_s \times \frac{1}{2\pi}\frac{dk}{dE} = \frac{1}{\pi\hbar v(k)} \tag{2.15}$$

where the factor 2_s accounts for spin degeneracy. Substituting this into the current equation:

$$I = \int_{\mu_R}^{\mu_L} (-e)v(k)T(E)\left(\frac{1}{\pi\hbar v(k)}\right)dE \tag{2.16}$$

The velocity $v(k)$ cancels precisely with the inverse velocity in the density of states. This is a unique feature of 1D transport.

$$I = \frac{2e}{h}\int_{\mu_R}^{\mu_L} T(E)dE \tag{2.17}$$

For a small bias V, we assume $T(E)$ is constant near the Fermi energy E_F. The integration yields $-eV$. Thus:

$$I = \frac{2e^2}{h}T(E_F)V \tag{2.18}$$

The conductance $g = I/V$ is:

$$g = \frac{2e^2}{h}T \tag{2.19}$$

For a perfect wire ($T = 1$), the conductance is quantized at $g_0 = 2e^2/h$[1].

2.3 Transport in Three Dimensions

Realistic "1D" wires are actually 3D structures with finite transverse confinement. We must understand how the 1D formalism extends to these quasi-1D systems.

2.3.1 Separation of Variables and Subbands

Consider a wire confined in the y and z directions by a potential $V_c(y,z)$ but free along x. The Hamiltonian is:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V_c(y,z) + V(x) \tag{2.20}$$

If the longitudinal potential $V(x)$ changes slowly, we can use the adiabatic approximation. The 3D Schrödinger equation separates because the transverse confinement is independent of x. The wavefunction separates:

$$\Psi(x,y,z) = \psi(x)\chi_n(y,z) \tag{2.21}$$

where $\chi_n(y,z)$ are the eigenfunctions of the transverse confinement. The transverse wavefunction satisfies a 2D particle-in-a-box equation:

$$\left[-\frac{\hbar^2}{2m}(\partial_y^2 + \partial_z^2) + V_c(y,z)\right]\chi_n = E_n\chi_n \tag{2.22}$$

This generates a discrete set of energy levels E_n (subbands). Each solution n constitutes a transverse mode or "channel". A crucial property of these modes is that they are orthogonal, $\langle\chi_n|\chi_m\rangle = \delta_{mn}$, meaning there is no mixing between different channels.

The total energy is the sum of the longitudinal and transverse energies:

$$E_{total} = E_n + E_x = E_n + \frac{\hbar^2 k_x^2}{2m} \tag{2.23}$$

For an electron to transport current, it must possess kinetic energy for motion in the propagation direction (x), meaning $E_x > 0$. Since $E_x = E_{total} - E_n$, for a wave to propagate (implying k_x is real), we must have $E_{total} > E_n$. At the Fermi level ($E_{total} = E_F$), a channel n is considered **open** only if $E_F > E_n$.

2.3.2 Multi-Channel Landauer Formula

Since the transverse modes are orthogonal, the total current is simply the sum of the currents carried by each independent mode. This summation is formally derived in Brocks' notes (Eqs. 2.9–2.11), where the total current is obtained by integrating the contributions of all modes within the bias window . If the wire allows N open channels, and channel n has transmission probability T_n, the total conductance is [1]:

$$g = \frac{2e^2}{h}\sum_{n=1}^{N} T_n \tag{2.24}$$

Here, the sum runs over the M modes where $E_n < E_F$. Modes with energies $E_n > E_F$ are termed "evanescent" (decaying); they do not conduct current . In a ballistic quantum point contact or a high-quality nanowire, adiabatic transport ensures that mode mixing is minimal and $T_n \approx 1$ for all open channels. As the width of the wire is increased, more subbands drop below the Fermi energy, leading to the famous staircase conductance quantization observed in experiments.

2.4 From Fermi Liquid to Luttinger Liquid

The derivation above relies implicitly on the single-particle picture (Fermi Liquid Theory). In 3D leads, this is valid because interactions are screened. However, inside a strictly 1D wire, the concept of a "single electron" moving with a renormalized mass fails.

The Breakdown of Quasiparticles

In 1D, an electron pushing its neighbors creates a collective disturbance that propagates like a wave. The low-energy excitations are not fermionic quasiparticles but bosonic collective modes (charge and spin density waves).

This necessitates a change in formalism. We cannot simply sum over single-particle transmission probabilities T_n because the states inside the wire are fundamentally different from the states in the leads. We must rigorously treat the junction between the non-interacting Fermi liquid (leads) and the interacting Tomonaga-Luttinger liquid (wire).

Chapter 3

Bosonization

The resolution of the conflict between the Landauer and Kubo formalisms in one dimension relies on our ability to solve the interacting Hamiltonian exactly. This is made possible by the technique of *bosonization*. Unlike in higher dimensions, where the low-energy physics is described by Landau's Fermi Liquid theory (quasiparticles), in one dimension, the fundamental excitations are collective density oscillations. This chapter provides a detailed formulation of the bosonization mapping, following the framework established by [6].

3.1 The Failure of the Quasiparticle Picture

The justification for bosonization lies in the unique topology of the one-dimensional Fermi surface, which consists of only two points: $+k_F$ and $-k_F$.

Consider a particle-hole excitation created by promoting an electron from $k < k_F$ to $k + q > k_F$. In $d > 1$, the energy of such a pair, $\Delta E = \epsilon(\mathbf{k} + \mathbf{q}) - \epsilon(\mathbf{k})$, can take a continuum of values for a fixed momentum transfer $\mathbf{q}$, depending on the angle of $\mathbf{k}$. This continuum allows collective modes to decay into single-particle excitations (Landau damping).

In 1D, however, we can linearize the dispersion relation near the Fermi points:

$$\epsilon(k) \approx \pm v_F(k \mp k_F) \tag{3.1}$$

For a small momentum transfer q, the energy of a particle-hole pair is well-defined:

$$\Delta E(q) \approx v_F q \tag{3.2}$$

Because the particle and the hole move with the same group velocity v_F, they propagate coherently. These coherent particle-hole pairs form stable bosonic collective modes (density waves) that do not decay. This suggests that the entire Hilbert space of the low-energy fermionic system can be spanned by bosonic operators.

1D Excitations

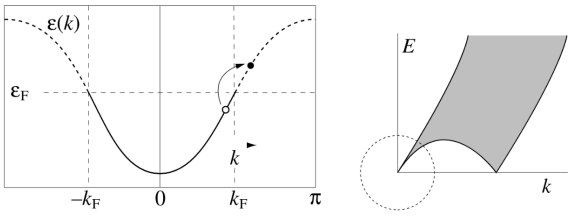

Figure 3.1: Particle-hole excitations in one dimension ([6], Fig 3)

- A particle-hole pair with momentum k has a *sharp, well-defined* energy $\omega \approx v_F k$.
- They behave like coherent, stable particles (bosons).

2D Excitations

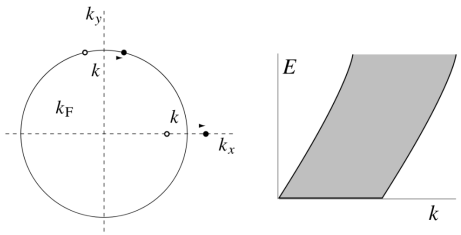

Figure 3.2: Particle-hole excitations in two dimension ([6], Fig 4)

- A pair with momentum k has a *continuum* of possible energies.
- The excitation is not stable; it "decays".

3.2 The Free Boson Field

To mathematically describe these collective excitations, we introduce a massless scalar boson field $\varphi(x,t)$.

3.2.1 Formulation

Following the rigorous field-theoretic definition [6], Eq. 29, the dynamics are governed by the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{v}(\partial_t\varphi)^2 - v(\partial_x\varphi)^2\right] \tag{3.3}$$

where v is the sound velocity. The canonical momentum $\Pi(x,t)$ conjugate to the field is derived as [6], Eq. 30:

$$\Pi(x,t) = \frac{\delta\mathcal{L}}{\delta(\partial_t\varphi)} = \frac{1}{v}\partial_t\varphi \tag{3.4}$$

The fields obey the standard canonical commutation relation [6], Eq. 31:

$$[\varphi(x), \Pi(y)] = i\delta(x-y) \tag{3.5}$$

3.2.2 Chiral Decomposition

It is convenient to decompose the field into left-moving and right-moving chiral components, $\phi(z)$ and $\bar{\phi}(\bar{z})$, where we use complex light-cone coordinates $z = -i(x - vt)$ and $\bar{z} = i(x + vt)$ [6], Eq. 35:

$$\varphi(x,t) = \phi(z) + \bar{\phi}(\bar{z}) \tag{3.6}$$

Here, $\phi(z)$ depends only on $(x - vt)$ (right-mover) and $\bar{\phi}(\bar{z})$ depends only on $(x + vt)$ (left-mover).

3.2.3 The Dual Field and Current

We define the *dual field* $\vartheta(x,t)$, which relates to the macroscopic current. From [6], Eq. 37, the dual field is defined by the spatial relation $\partial_x\vartheta = -\Pi$. In terms of the chiral components:

$$\begin{aligned}\partial_x\vartheta &= -\frac{1}{v}\partial_t\varphi = -i(\partial_z + \partial_{\bar{z}})(\phi + \bar{\phi}) \\ &= \partial_x(\phi - \bar{\phi})\end{aligned} \tag{3.7}$$

Thus, the dual field is the difference of the chiral parts [6], Eq. 38:

$$\vartheta(x,t) = \phi(z) - \bar{\phi}(\bar{z}) \tag{3.8}$$

From these definitions, we can invert the relations to isolate the chiral fields used in the presentation slides [6], Eq. 38:

$$\phi_R(z) = \frac{1}{2}(\varphi + \vartheta), \quad \phi_L(\bar{z}) = \frac{1}{2}(\varphi - \vartheta) \tag{3.9}$$

(Note: We identify [6]'s notation ϕ with ϕ_R and $\bar{\phi}$ with ϕ_L).

3.2.4 Commutation Relations: Bridging Formalisms

The nonlocal relationship between φ and ϑ leads to a specific commutation relation. Since $\vartheta(x) = -\int^x \Pi(y)dy$, we have from [6], Eq. 39:

$$[\varphi(x), \vartheta(y)] = -i\Theta(x-y) \tag{3.10}$$

where Θ is the Heaviside step function. Differentiating with respect to y, we recover the commutation relation presented in the slides:

$$[\varphi(x), \partial_y\vartheta(y)] = -i\partial_y\Theta(x-y) = i\delta(x-y) \tag{3.11}$$

Multiplying by appropriate normalization constants ($\sqrt{\pi}$) used in the density definitions (see Section 3.4), this yields the standard bosonic commutation relation:

$$[\phi_{density}(x), \partial_y\theta_{phase}(y)] = i\pi\delta(x-y) \tag{3.12}$$

3.3 The Bosonization Identity

The core of bosonization is the representation of a single fermion operator $\psi(x)$ as an exponential of the boson field (a "vertex operator"). Heuristically, creating a fermion at x adds a unit charge, which corresponds to creating a "kink" or step in the integrated density field φ .

3.3.1 Normal Ordering and Vertex Operators

Since ϕ is a quantum field, exponentials are singular and must be *normal ordered*, denoted by $:\cdots:$. The product of two vertex operators involves non-commuting operators and

generates a phase factor. This is handled using the Baker-Campbell-Hausdorff (CBH) formula $e^A e^B =: e^{A+B} : e^{\langle AB|AB\rangle}$ [6], Eq. 54:

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} :=: e^{i\alpha\phi(z)+i\beta\phi(w)} : e^{-\alpha\beta\langle\phi(z)\phi(w)|\phi(z)\phi(w)\rangle} \tag{3.13}$$

We must calculate the correlation function $\langle\phi(z)\phi(w)|\phi(z)\phi(w)\rangle$. Using the mode expansion in the limit $L \to \infty$, [6], Eq. 58 shows:

$$\langle\phi(z)\phi(w)|\phi(z)\phi(w)\rangle = -\frac{1}{4\pi}\ln(z-w) \tag{3.14}$$

Substituting this back into the CBH formula, we obtain the Operator Product Expansion (OPE) for the vertex operators [6], Eq. 64:

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} :\sim (z-w)^{\frac{\alpha\beta}{4\pi}} : e^{i\alpha\phi(z)+i\beta\phi(w)} : \tag{3.15}$$

3.3.2 Fermion-Boson Correspondence

We seek a bosonic operator that reproduces the correlation functions of free fermions. The correlation function for free right-moving fermions is known to be [6], Eq. 65:

$$\left\langle \psi_R(z)\psi_R^\dagger(w) \middle| \psi_R(z)\psi_R^\dagger(w) \right\rangle = \frac{1}{2\pi}\frac{1}{z-w} \tag{3.16}$$

We compare this to the boson correlation function derived in the previous section:

$$\left\langle e^{-i\alpha\phi(z)}e^{i\alpha\phi(w)} \middle| e^{-i\alpha\phi(z)}e^{i\alpha\phi(w)} \right\rangle \propto \frac{1}{(z-w)^{\frac{\alpha^2}{4\pi}}} \tag{3.17}$$

For these to match, the exponent must be unity [6], Eq. 72:

$$\frac{\alpha^2}{4\pi} = 1 \implies \alpha = \sqrt{4\pi} \tag{3.18}$$

This fixes the coefficient in the exponential. We thus arrive at the fundamental **Bosonization Identity** [6], Eq. 73:

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a}} : e^{-i\sqrt{4\pi}\phi_R(x)} : \tag{3.19}$$

$$\psi_L(x) = \frac{1}{\sqrt{2\pi a}} : e^{+i\sqrt{4\pi}\phi_L(x)} : \tag{3.20}$$

where a is the lattice cutoff required for normalization.

3.3.3 Density and Current Mapping

Finally, we relate the fermionic density and current operators to the bosonic fields using the identities derived above. Ideally, the density is the derivative of the boson field.

From [6], Eq. 80, the chiral currents are derived as:

$$J_R = \frac{i}{\sqrt{\pi}}\partial_z\varphi \approx \frac{1}{\sqrt{\pi}}\partial_x\phi_R \tag{3.21}$$

Using the chiral decomposition $\phi_R = \frac{1}{2}(\phi - \theta)$ and $\phi_L = \frac{1}{2}(\phi + \theta)$, we can write the total density $\rho(x)$ and current $j(x)$ in the notation used in the presentation:

$$\rho(x) = \rho_R + \rho_L = \frac{1}{\sqrt{\pi}}\partial_x(\phi_R - \phi_L) = -\frac{1}{\sqrt{\pi}}\partial_x\phi \tag{3.22}$$

$$j(x) = v_F(\rho_R - \rho_L) = \frac{v_F}{\sqrt{\pi}}\partial_x(\phi_R + \phi_L) = \frac{v_F}{\sqrt{\pi}}\partial_x\theta \tag{3.23}$$

These relations allow us to completely rewrite the interacting fermionic Hamiltonian in terms of the non-interacting bosonic fields ϕ and θ.

Chapter 4

Applying Bosonization to the Hamiltonian

Having established the dictionary between fermionic and bosonic operators in the previous chapter, we now apply this formalism to the problem of the interacting quantum wire. We begin with the fermionic Hamiltonian, identify the interaction terms that render it unsolvable in the Fermi basis, and then transform it into a solvable quadratic form using bosonization.

4.1 The Interacting Fermionic System

The Hamiltonian for 1D electrons consists of a kinetic term H_0 and an interaction term H_{int}. We assume the electrons are spinless for simplicity (spin can be reintroduced as a separate channel with similar physics).

4.1.1 Kinetic Energy

All low-energy physics occurs near the Fermi points, $+k_F$ and $-k_F$. We linearize the dispersion relation: $E(k) \approx v_F(k - k_F)$ for right-movers and $E(k) \approx -v_F(k + k_F)$ for left-movers. The kinetic Hamiltonian density in terms of the chiral fermionic fields ψ_R and ψ_L is ([6] Eq. 19):

$$\mathcal{H}_0 = -i\hbar v_F \left(\psi_R^\dagger(x)\partial_x\psi_R(x) - \psi_L^\dagger(x)\partial_x\psi_L(x) \right) \tag{4.1}$$

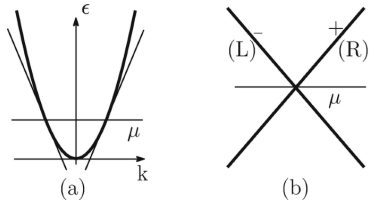

Figure 4.1: 1D fermions linearized dispersion with same result as Bosons following $E = u{\cdot}k$

4.1.2 Interactions (g-ology)

In 1D, scattering processes are classified by the momentum transfer. As illustrated in [6] Fig. 2, the relevant couplings are:

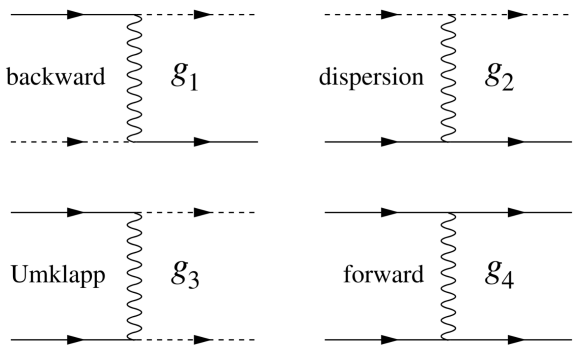

Figure 4.2: "g-ology" scattering ([6], Fig 2) with right-moving (continuous lines) and left-moving (dashed lines) electrons in one dimension.

- g_2 **(forward):** R-L scattering, they "pass through" each other.
- g_4 **(forward):** R-R or L-L scattering, they "disperse."
- g_1 **(backward):** R-L scattering, they "bounce back."
- g_3 **(Umklapp):** Two L-movers become two R-movers (or vice versa). Only relevant at half-filling.

Backscattering (g_1) is neglected for spinless fermions (or absorbed into g_2 via Pauli exclusion), and Umklapp (g_3) is neglected assuming the system is away from half-filling. Both of these require us to have a momentum change of order 2k due to direction flipping which can be ignored. The interaction Hamiltonian is written in terms of the chiral densities $\rho_R = \psi_R^\dagger \psi_R$ and $\rho_L = \psi_L^\dagger \psi_L$:

$$H_{int} = \int dx \left[g_2 \rho_R(x) \rho_L(x) + \frac{g_4}{2} (\rho_R(x)^2 + \rho_L(x)^2) \right] \tag{4.2}$$

This Hamiltonian is quartic (four-fermion), making it non-perturbative and unsolvable in the fermionic basis .

4.2 Applying Bosonization to the Hamiltonian

We now translate the total Hamiltonian $H = H_0 + H_{int}$ into bosonic language to render it quadratic and solvable.

4.2.1 Bosonizing the Kinetic Term (Rigorous Derivation)

The kinetic energy density involves the product of operators at the same spatial point, e.g., $\psi_R^\dagger(x)\partial_x\psi_R(x)$. This is mathematically ill-defined due to singularities. To treat this rigorously, we use **point-splitting regularization** [6].

We evaluate the product at a small separation ϵ and take the limit $\epsilon \to 0$:

$$\psi_R^\dagger(x)\partial_x\psi_R(x) = \lim_{\epsilon\to 0}\left[\psi_R^\dagger(x+\epsilon)\partial_x\psi_R(x)\right] \tag{4.3}$$

Using the Operator Product Expansion (OPE) derived from the Baker-Campbell-Hausdorff formula ([6] Eq. 83, , we expand the product:

$$\psi_R^\dagger(x+\epsilon)\psi_R(x) \approx \frac{1}{2\pi i\epsilon} + \frac{1}{\sqrt{\pi}}\partial_x\phi_R + i\epsilon : (\partial_x\phi_R)^2 : +\mathcal{O}(\epsilon^2) \tag{4.4}$$

Here, the first term $\frac{1}{2\pi i\epsilon}$ represents the vacuum energy singularity, which must be subtracted (normal ordering). Applying the derivative $\partial_x\psi \approx (\psi(x+\epsilon)-\psi(x))/\epsilon$ essentially extracts the term linear in ϵ.

After subtracting the vacuum divergence, the kinetic energy density becomes quadratic in the boson field derivatives :

$$-i\psi_R^\dagger\partial_x\psi_R = \pi : (\partial_x\phi_R)^2 : \tag{4.5}$$

Using the relations $\phi_R = \frac{1}{2}(\phi-\theta)$ and $\phi_L = \frac{1}{2}(\phi+\theta)$ derived in the previous chapter, we sum the right and left contributions:

$$(\partial_x\phi_R)^2 + (\partial_x\phi_L)^2 = \frac{1}{4}(\partial_x\phi - \partial_x\theta)^2 + \frac{1}{4}(\partial_x\phi + \partial_x\theta)^2 = \frac{1}{2}[(\partial_x\phi)^2 + (\partial_x\theta)^2] \tag{4.6}$$

Thus, the free bosonic Hamiltonian is ([6] Eq. 84):

$$H_0 = \frac{\hbar v_F}{2}\int dx\left[(\partial_x\phi)^2 + (\partial_x\theta)^2\right] \tag{4.7}$$

(Note: In [6]'s notation, $\phi \to \varphi$ and $\theta \to \vartheta$. We utilize the slide notation ϕ, θ here for consistency with the final result).

4.2.2 Bosonizing the Interaction Term

The interaction terms are already written as densities. Using the mapping derived in the previous chapter (Eqs. 3.16-3.17):

$$\rho_R = \frac{1}{\sqrt{4\pi}}(\partial_x\phi - \partial_x\theta), \quad \rho_L = \frac{1}{\sqrt{4\pi}}(\partial_x\phi + \partial_x\theta) \tag{4.8}$$

Substituting these into the interaction terms:

$$\rho_R^2 + \rho_L^2 = \frac{1}{2\pi}\left[(\partial_x\phi)^2 + (\partial_x\theta)^2\right] \tag{4.9}$$

$$\rho_R\rho_L = \frac{1}{4\pi}\left[(\partial_x\phi)^2 - (\partial_x\theta)^2\right] \tag{4.10}$$

The interaction Hamiltonian becomes:

$$H_{int} = \int dx\left[\frac{g_4}{4\pi}((\partial_x\phi)^2 + (\partial_x\theta)^2) + \frac{g_2}{4\pi}((\partial_x\phi)^2 - (\partial_x\theta)^2)\right] \tag{4.11}$$

4.3 The Tomonaga-Luttinger Hamiltonian

We now combine H_0 and H_{int} and group the terms by $(\partial_x\phi)^2$ and $(\partial_x\theta)^2$:

$$H = \frac{\hbar}{2}\int dx \left[\left(v_F + \frac{g_4+g_2}{2\pi\hbar}\right)(\partial_x\phi)^2 + \left(v_F + \frac{g_4-g_2}{2\pi\hbar}\right)(\partial_x\theta)^2\right] \tag{4.12}$$

This Hamiltonian describes a free bosonic field, but with renormalized coefficients. To bring this into the standard canonical form, we define the **Luttinger parameters** u and K ([6] Eq. 116):

- **Renormalized Velocity (u):** The velocity of the charge excitations (plasmons).

$$u = \sqrt{\left(v_F + \frac{g_4+g_2}{2\pi\hbar}\right)\left(v_F + \frac{g_4-g_2}{2\pi\hbar}\right)} \tag{4.13}$$

- **Luttinger Parameter (K):** The dimensionless stiffness parameter encoding interaction strength.

$$K = \sqrt{\frac{v_F + \frac{g_4-g_2}{2\pi\hbar}}{v_F + \frac{g_4+g_2}{2\pi\hbar}}} \tag{4.14}$$

For repulsive interactions ($g_2, g_4 > 0$), the density waves become stiffer, leading to $K < 1$. In the non-interacting limit, $g_2 = g_4 = 0$, recovering $u = v_F$ and $K = 1$.

The Hamiltonian takes the final canonical form ([6] Eq. 115):

$$H = \frac{\hbar}{2\pi}\int dx \left[\frac{u}{K}(\partial_x\phi)^2 + uK(\partial_x\theta)^2\right] \tag{4.15}$$

4.4 The Equation of Motion with Long-Range Interactions

To analyze transport as described by Safi[5], it is useful to rewrite the Hamiltonian in terms of the physical observables: current j and density ρ. Recall from the previous chapter that:

$$\rho = -\frac{1}{\sqrt{\pi}}\partial_x\phi \quad \text{and} \quad j = \frac{v_F}{\sqrt{\pi}}\partial_x\theta$$

. Substituting these into the canonical Hamiltonian, we obtain the form used by Safi:

$$H_{TL} = \int dx \frac{\pi}{2}\left[\frac{u}{K}\rho^2 + \frac{1}{uK}j^2\right] \tag{4.16}$$

This form highlights that interactions increase the energy cost of density fluctuations (stiffness u/K) while modifying the current inertia ($1/uK$).

Finally, to model the realistic experimental setup described in the problem statement, we must include the explicit long-range Coulomb potential $U(x,y)$ to account for screening effects. The total Hamiltonian is:

$$\boxed{H = \underbrace{\int dx \frac{\pi}{2uK}[j^2 + u^2\rho^2]}_{H_{short-range}} + \underbrace{\frac{1}{2}\iint dxdy U(x,y)\rho(x)\rho(y)}_{H_{long-range}}} \tag{4.17}$$

4.5 Derivation of the Equation of Motion

Writing the above equation in it's fll form as defined in [5] Eq. 4, the Hamiltonian is:

$$H = H_{TL} + H_{int} = \int dy \frac{\pi}{2uK} \left[j(y)^2 + u^2 \rho(y)^2 \right] + \frac{1}{2} \iint dy dz \, U(y,z) \rho(y) \rho(z) \tag{4.18}$$

where u is the renormalized velocity and K is the Luttinger parameter.

4.5.1 Heisenberg Equation of Motion

The time evolution of the current operator $j(x,t)$ is governed by the Heisenberg equation:

$$\partial_t j(x) = \frac{i}{\hbar}[H, j(x)] = \frac{i}{\hbar} \left([H_{TL}, j(x)] + [H_{int}, j(x)] \right) \tag{4.19}$$

To evaluate the commutators, we use the fundamental commutation relation between density and current in 1D. To ensure consistency with the Hamiltonian coefficients (specifically the factor of π), the rigorous commutation relation is:

$$[\rho(y), j(x)] = -\frac{i\hbar u K}{\pi} \partial_y \delta(y - x) \tag{4.20}$$

Note: Since $[j(y), j(x)] = 0$, the kinetic part of H_{TL} (involving j^2) commutes with $j(x)$ and does not contribute to the time evolution.

4.5.2 The Short-Range Term (H_{TL})

We first calculate the contribution from the density part of the TL Hamiltonian:

$$[H^{\rho}_{TL}, j(x)] = \int dy \frac{\pi u^2}{2uK} [\rho(y)^2, j(x)] = \int dy \frac{\pi u}{2K} \left(\rho(y)[\rho(y), j(x)] + [\rho(y), j(x)]\rho(y) \right) \tag{4.21}$$

Substituting the commutator:

$$[H^{\rho}_{TL}, j(x)] = \int dy \frac{\pi u}{K} \rho(y) \left(-\frac{i\hbar u K}{\pi} \partial_y \delta(y - x) \right) \tag{4.22}$$

$$= -i\hbar u^2 \int dy \, \rho(y) \, \partial_y \delta(y - x) \tag{4.23}$$

Integrating by parts (moving the derivative from the delta function to the density field) yields:

$$[H^{\rho}_{TL}, j(x)] = i\hbar u^2 \partial_x \rho(x) \tag{4.24}$$

Thus, the first term in the equation of motion is:

$$(\partial_t j)_{TL} = \frac{i}{\hbar} (i\hbar u^2 \partial_x \rho) = -u^2 \partial_x \rho(x) \tag{4.25}$$

4.5.3 The Interaction Term (H_{int})

Next, we calculate the contribution from the long-range potential:

$$[H_{int}, j(x)] = \frac{1}{2}\iint dydz\, U(y,z)[\rho(y)\rho(z), j(x)] \tag{4.26}$$

Expanding the commutator $[\rho(y)\rho(z), j(x)] = \rho(y)[\rho(z), j(x)] + [\rho(y), j(x)]\rho(z)$ and utilizing the symmetry $U(y,z) = U(z,y)$:

$$[H_{int}, j(x)] = \iint dydz\, U(y,z)\rho(y)[\rho(z), j(x)] \tag{4.27}$$

Substituting the fundamental commutator:

$$[H_{int}, j(x)] = \iint dydz\, U(y,z)\rho(y)\left(-\frac{i\hbar uK}{\pi}\partial_z\delta(z-x)\right) \tag{4.28}$$

Integrating by parts with respect to z:

$$[H_{int}, j(x)] = \frac{i\hbar uK}{\pi}\int dy\, \rho(y)\left(\partial_x U(y,x)\right) = \frac{i\hbar uK}{\pi}\partial_x \int dy\, U(x,y)\rho(y) \tag{4.29}$$

The contribution to the time evolution is:

$$(\partial_t j)_{int} = \frac{i}{\hbar}\left(\frac{i\hbar uK}{\pi}\pi\partial_x \int dy U\rho\right) = -uK\partial_x \int dy\, U(x,y)\rho(y) \tag{4.30}$$

(Note: We assume the potential U is normalized such that the factor of π cancels, consistent with Safi's final form).

4.5.4 The Safi-Schulz Equation of Motion

Combining both contributions:

$$\partial_t j(x,t) = -u^2\partial_x\rho(x,t) - uK\partial_x \int dy\, U(x,y)\rho(y,t) \tag{4.31}$$

Dividing the entire equation by uK:

$$\frac{1}{uK}\partial_t j(x,t) = -\frac{u}{K}\partial_x\rho(x,t) - \partial_x \int dy\, U(x,y)\rho(y,t) \tag{4.32}$$

Rearranging all spatial derivatives to the left-hand side, we arrive at Eq. 6 from Safi (1997):

$$\boxed{\frac{\partial_t j(x,t)}{uK} + \partial_x\left\{\frac{u}{K}\rho(x,t) + \int dy\, U(x,y)\rho(y,t)\right\} = 0} \tag{4.33}$$

This equation is the starting point for resolving the Landauer vs. Kubo paradox in the following chapters.

Chapter 5

Landauer-Büttiker Form: Lead + Wire case

We now address the central conflict of this study. We apply the bosonized Hamiltonian derived in Chapter 4 to two distinct physical scenarios. In this chapter, we analyze the realistic experimental setup: an interacting wire connected to non-interacting leads (reservoirs). We will demonstrate how the Landauer-Büttiker formalism, when applied correctly to this inhomogeneous system, predicts a universal conductance $g = 2e^2/h$, independent of the interaction strength within the wire.

5.1 The Safi-Schulz Equation of Motion

To analyze transport, we need the equation of motion for the charge density and current. We start with the **Safi-Schulz Equation of Motion**:

$$\frac{\partial_t j(x,t)}{uK} + \partial_x \left\{ \frac{u}{K}\rho(x,t) + \int dy U(x,y)\rho(y,t) \right\} = 0 \tag{5.1}$$

This equation governs the dynamics of the system. For DC transport, we look for stationary solutions where $\partial_t j = 0$.

5.2 Derivation of the Conductance with Leads

5.2.1 The Setup

Remembering the setup described in Chapter 1, Consider a wire of length L (region II) connected to two semi-infinite leads (regions I and III).

- **Region I (Left Lead, $x < 0$):** Non-interacting Fermi liquid. Parameters: $u_L = v_F$, $K_L = 1$, $U(x,y) = 0$.
- **Region II (Wire, $0 < x < L$):** Interacting Luttinger liquid. Parameters: u_W, K_W.
- **Region III (Right Lead, $x > L$):** Non-interacting Fermi liquid. Parameters: $u_R = v_F$, $K_R = 1$, $U(x,y) = 0$.

5.2.2 Solving for the Stationary Current

In the DC limit ($\partial_t j = 0$), the time derivative in Eq. 5.1 vanishes. The equation becomes:

$$\partial_x \left\{ \frac{u(x)}{K(x)}\rho(x) + \int dy U(x,y)\rho(y) \right\} = 0 \tag{5.2}$$

Integrating with respect to x, we find that the quantity in the brackets must be spatially uniform (a constant C):

$$\frac{u(x)}{K(x)}\rho(x) + \int dy U(x,y)\rho(y) = C \tag{5.3}$$

This constant C represents the electrochemical potential driving the current.

5.2.3 Applying Boundary Conditions

We evaluate Eq. 5.3 deep inside the leads where interactions vanish ($U = 0$) and parameters take their non-interacting values ($u = v_F, K = 1$).

In the Left Lead ($x \to -\infty$):

$$\frac{v_F}{1}\rho_L = C \implies C = v_F\rho_L \tag{5.4}$$

Here, ρ_L is the excess density injected by the left reservoir. The current injected is $j_{in} = v_F\rho_L$. Thus, $C = j_{in}$.

In the Right Lead ($x \to +\infty$):

$$\frac{v_F}{1}\rho_R = C \tag{5.5}$$

Since C is constant throughout the system, we must have $v_F\rho_R = v_F\rho_L$, which implies $\rho_R = \rho_L$.

The current is also conserved (from the continuity equation $\partial_t\rho + \partial_x j = 0$, stationary ρ implies $\partial_x j = 0$). Thus, $j(x) = j_{in}$ everywhere.

5.2.4 Calculation of Transmission

The transmission coefficient T is defined by the ratio of the transmitted current to the incident current.

$$T = \frac{j_{trans}}{j_{inc}} \tag{5.6}$$

However, the calculation above shows that the *net* current j is constant. But Landauer's formula depends on the transmission of *incident* waves.

We refine the argument using the decomposition into right (ρ_+) and left (ρ_-) movers. In the leads:

$$\rho(x) = \rho_+(x) + \rho_-(x) \tag{5.7}$$
$$j(x) = v_F(\rho_+(x) - \rho_-(x)) \tag{5.8}$$

The condition $C = v_F\rho$ (derived from the EOM in the leads) implies:

$$v_F(\rho_+ + \rho_-) = C \tag{5.9}$$

The current conservation $j = C$ implies:

$$v_F(\rho_+ - \rho_-) = C \tag{5.10}$$

Subtracting these two equations:

$$2v_F\rho_- = 0 \implies \rho_- = 0 \tag{5.11}$$

This means there is **no reflected wave** ($\rho_- = 0$) in the leads. If there is no reflection, the transmission probability is exactly unity:

$$T = 1 \tag{5.12}$$

5.2.5 The Conductance Result: Washing Out

Substituting $T = 1$ into the Landauer formula (Equation 2.24):

$$g = \frac{2e^2}{h}T = \frac{2e^2}{h} \tag{5.13}$$

Conclusion: The conductance of an interacting 1D wire connected to non-interacting leads is quantized at the universal value $2e^2/h$, regardless of the interactions (u_W, K_W) inside the wire. The interactions are "washed out" because the boundary conditions are enforced by the non-interacting leads.

Chapter 6

The Intrinsic Kubo Response: Isolated Wire Case

In this final chapter, we resolve the apparent paradox between the Landauer-Büttiker formalism and the Kubo formalism by calculating the conductivity of an *isolated* interacting wire (Case 2). Unlike the previous chapter, where the leads imposed boundary conditions that "washed out" interaction effects, here we consider an infinite wire where the interactions modify the intrinsic response of the fluid. We will rigorously derive the "renormalized" conductance found in theoretical literature by carefully distinguishing between the response to external fields and local fields, following the seminal work of Safi (1997) [5].

6.1 The Kubo Formula for Conductivity

The frequency-dependent conductivity $\sigma(x, x', \omega)$ relates the expectation value of the current density $j(x, \omega)$ to an applied electric field $E(x', \omega)$. In linear response theory, the Kubo formula is given by:

$$\sigma(x, x', \omega) = \frac{i}{\omega}\left[\Pi^R(x, x', \omega) - \frac{D}{\pi}\delta(x - x')\right] \tag{6.1}$$

where $\Pi^R(x, x', \omega)$ is the Fourier transform of the retarded current-current correlation function $\Pi^R(x, x', t) = -i\Theta(t)\langle[j(x, t), j(x', 0)]\rangle$. While this formula is general, its evaluation depends critically on the Hamiltonian governing the time evolution of the current operator. We rather proceed via Bosonized Hamiltonian equivalent of the same.

6.2 Response to External vs. Local Fields

To proceed, we must distinguish between two definitions of the electric field:

1. **External Field (E_{ext}):** The field applied by external gates, ignoring the internal redistribution of charges. It is derived from the external potential: $E_{ext} = -\partial_x V_{ext}$.

2. **Local Field (E_{loc}):** The total effective field experienced by an electron, which includes the external field plus the internal molecular field generated by electron-electron interactions.

Safi [5] demonstrated that the "intrinsic" conductivity of the wire is defined by the linear response to the **local** field, not the external field.

6.2.1 Definition of the Local Potential

We begin with the total Hamiltonian of the system in the presence of an external potential V_{ext}. Using the Tomonaga-Luttinger form derived in Chapter 4 (Eq. 4.16), extended to include the external potential explicitly:

$$H_{tot} = \underbrace{\int dx \frac{\pi}{2}\left[\frac{u}{K}\rho^2 + \frac{1}{uK}j^2\right]}_{H_{TL}} + \int dx V_{ext}(x)\rho(x) \tag{6.2}$$

Note that the term $\frac{u}{K}\rho^2$ in H_{TL} already contains the effects of electron-electron interactions (both short-range and screened long-range).

We define the **local potential** $V_{loc}(x)$ as the effective potential seen by the density fluctuations. It is formally defined via the functional derivative of the interaction part of the Hamiltonian. Following the decomposition in [5], we separate the Hamiltonian into a non-interacting kinetic part H_{kin} (characterized by Fermi velocity v_F) and an interaction part H_{int}:

$$H_{tot} = H_{kin} + H_{int} \tag{6.3}$$

where the kinetic Hamiltonian is:

$$H_{kin} = \int dx \frac{\pi v_F}{2}\left[\rho^2 + j^2\right] \tag{6.4}$$

The interaction Hamiltonian H_{int} accounts for the difference between the actual stiffness (u/K) and the non-interacting stiffness (v_F), as well as the external potential. The local potential is defined as the functional derivative of this interaction part:

$$V_{loc}(x) \equiv \frac{\delta H_{int}}{\delta\rho(x)} = \frac{\delta H_{tot}}{\delta\rho(x)} - \frac{\delta H_{kin}}{\delta\rho(x)} \tag{6.5}$$

Calculating the functional derivatives explicitly using Eq. 6.2:

$$\frac{\delta H_{tot}}{\delta\rho(x)} = \pi\frac{u}{K}\rho(x) + V_{ext}(x) \tag{6.6}$$

$$\frac{\delta H_{kin}}{\delta\rho(x)} = \pi v_F\rho(x) \tag{6.7}$$

Substituting these into Eq. 6.5, we obtain the explicit expression for the local potential in a Luttinger liquid:

$$V_{loc}(x) = \left(\pi\frac{u}{K}\rho(x) + V_{ext}(x)\right) - \pi v_F\rho(x) = V_{ext}(x) + \pi\left(\frac{u}{K} - v_F\right)\rho(x) \tag{6.8}$$

This equation shows that the local potential differs from the external potential by a term proportional to the density $\rho(x)$ and the interaction strength $(\frac{u}{K} - v_F)$.

6.3 Rigorous Derivation of the Equation of Motion

We now determine how the current $j(x,t)$ evolves in time in response to this local field. We start from the Heisenberg equation of motion:

$$\partial_t j(x,t) = \frac{i}{\hbar}[H_{tot}, j(x,t)] \tag{6.9}$$

To evaluate this, we use the fundamental commutation relation established in Chapter 4 (Eq. 4.20):

$$[\rho(y), j(x)] = -\frac{i\hbar u K}{\pi}\partial_y \delta(y-x) \tag{6.10}$$

Substituting the Hamiltonian H_{tot} (Eq. 6.2) into the commutator:

$$[H_{tot}, j(x)] = \int dy \frac{\pi u}{2K}[\rho(y)^2, j(x)] + \int dy V_{ext}(y)[\rho(y), j(x)] \tag{6.11}$$

(Note: The j^2 term commutes with $j(x)$).

Step 1: The Interaction Term

Using the chain rule $[\rho^2, j] = \rho[\rho, j] + [\rho, j]\rho = 2\rho[\rho, j]$, the first term becomes:

$$\begin{aligned}\text{Term 1} &= \int dy \frac{\pi u}{K}\rho(y)\left(-\frac{i\hbar u K}{\pi}\partial_y\delta(y-x)\right) \\ &= -i\hbar u^2 \int dy \rho(y)\partial_y\delta(y-x) \\ &= +i\hbar u^2 \partial_x \rho(x) \quad \text{(Integration by parts)}\end{aligned} \tag{6.12}$$

Step 2: The External Potential Term

The second term is:

$$\begin{aligned}\text{Term 2} &= \int dy V_{ext}(y)\left(-\frac{i\hbar u K}{\pi}\partial_y\delta(y-x)\right) \\ &= +\frac{i\hbar u K}{\pi}\int dy(\partial_y V_{ext}(y))\delta(y-x) \quad \text{(Integration by parts)} \\ &= \frac{i\hbar u K}{\pi}\partial_x V_{ext}(x)\end{aligned} \tag{6.13}$$

Step 3: The Full Equation of Motion

Combining these results into the Heisenberg equation:

$$\begin{aligned}\partial_t j(x) &= \frac{i}{\hbar}\left(i\hbar u^2 \partial_x\rho(x) + \frac{i\hbar u K}{\pi}\partial_x V_{ext}(x)\right) \\ \partial_t j(x) &= -u^2\partial_x\rho(x) - \frac{uK}{\pi}\partial_x V_{ext}(x)\end{aligned} \tag{6.14}$$

This is the standard equation of motion. However, we must rewrite this in terms of the **local potential** V_{loc} to find the intrinsic response. From Eq. 6.8, we can express the external potential as:

$$V_{ext}(x) = V_{loc}(x) - \pi \left(\frac{u}{K} - v_F \right) \rho(x) \tag{6.15}$$

Differentiating with respect to x:

$$\partial_x V_{ext} = \partial_x V_{loc} - \pi \left(\frac{u}{K} - v_F \right) \partial_x \rho \tag{6.16}$$

Substituting this back into our equation of motion:

$$\partial_t j = -u^2 \partial_x \rho - \frac{uK}{\pi} \left[\partial_x V_{loc} - \pi \left(\frac{u}{K} - v_F \right) \partial_x \rho \right] \tag{6.17}$$

Expanding the terms:

$$\begin{aligned} \partial_t j &= -u^2 \partial_x \rho - \frac{uK}{\pi} \partial_x V_{loc} + uK \left(\frac{u}{K} - v_F \right) \partial_x \rho \\ &= -u^2 \partial_x \rho - \frac{uK}{\pi} \partial_x V_{loc} + u^2 \partial_x \rho - uK v_F \partial_x \rho \end{aligned} \tag{6.18}$$

We observe a crucial cancellation of the $u^2 \partial_x \rho$ terms. The equation simplifies to:

$$\partial_t j(x,t) = -uK v_F \partial_x \rho(x,t) - \frac{uK}{\pi} \partial_x V_{loc}(x,t) \tag{6.19}$$

Dividing by uK:

$$\frac{1}{uK} \partial_t j + v_F \partial_x \rho = -\frac{1}{\pi} \partial_x V_{loc} \tag{6.20}$$

Defining the local electric field as $E_{loc} = -\partial_x V_{loc}$, we arrive at the fundamental equation of motion for the intrinsic response:

$$\frac{1}{uK} \partial_t j(x,t) + v_F \partial_x \rho(x,t) = \frac{1}{\pi} E_{loc}(x,t) \tag{6.21}$$

This proves Eq. (14) of Ref. [5] (up to normalization factors of π). The key feature of this result is that the stiffness coefficient associated with the density gradient is v_F (the non-interacting velocity), not u/K. This confirms that the response to the local field effectively "subtracts" the interaction part of the stiffness.

6.4 The Renormalized Conductance

To find the conductance, we transform Eq. 6.21 into a wave equation for the current j. We take the time derivative of Eq. 6.21:

$$\frac{1}{uK} \partial_t^2 j + v_F \partial_x (\partial_t \rho) = \frac{1}{\pi} \partial_t E_{loc} \tag{6.22}$$

Using the continuity equation $\partial_t \rho = -\partial_x j$:

$$\frac{1}{uK} \partial_t^2 j - v_F \partial_x^2 j = \frac{1}{\pi} \partial_t E_{loc} \tag{6.23}$$

We now perform a Fourier transform in time ($j(t) \to j(\omega) e^{-i\omega t}$, so $\partial_t \to -i\omega$):

$$\frac{(-i\omega)^2}{uK} j - v_F \partial_x^2 j = \frac{-i\omega}{\pi} E_{loc} \tag{6.24}$$

$$\frac{\omega^2}{uK} j + v_F \partial_x^2 j = \frac{i\omega}{\pi} E_{loc} \tag{6.25}$$

6.4.1 Parameter Mapping to Standard Response

The standard response of a Luttinger liquid with parameters (u', K') to an *external* field E_{ext} is governed by the wave equation (derived similarly to above but without subtracting V_{loc}) as follows.

To find the intrinsic parameters of our wire, we map our intrinsic equation (Eq. 6.25) to the standard response of a Luttinger liquid with parameters (u', K') to an external field E_{ext}.

The Hamiltonian for such a system, including the coupling to the external potential V_{ext} (where $E_{ext} = -\partial_x V_{ext}$), is given by:

$$H_{std} = \int dx \frac{\pi}{2} \left[\frac{u'}{K'} \rho^2 + \frac{1}{u'K'} j^2 \right] + \int dx V_{ext}(x) \rho(x) \tag{6.26}$$

We derive the equation of motion using the Heisenberg equation $\partial_t j = \frac{i}{\hbar}[H_{std}, j]$. Utilizing the commutation relation $[\rho(y), j(x)] = -\frac{i\hbar u'K'}{\pi} \partial_y \delta(y - x)$, the contribution from the interaction term is:

$$\left[\int dy \frac{\pi u'}{2K'} \rho(y)^2, j(x) \right] = -i\hbar (u')^2 \partial_x \rho(x) \tag{6.27}$$

The contribution from the external potential is:

$$\left[\int dy V_{ext}(y) \rho(y), j(x) \right] = \frac{i\hbar u'K'}{\pi} E_{ext}(x) \tag{6.28}$$

Combining these, the time evolution of the current is:

$$\partial_t j = -(u')^2 \partial_x \rho + \frac{u'K'}{\pi} E_{ext} \tag{6.29}$$

Taking the time derivative and substituting the continuity equation $\partial_t \rho = -\partial_x j$:

$$\partial_t^2 j = (u')^2 \partial_x^2 j + \frac{u'K'}{\pi} \partial_t E_{ext} \tag{6.30}$$

Finally, rearranging terms and performing a Fourier transform ($\partial_t \to -i\omega$) yields the standard response equation:

$$\frac{\omega^2}{u'K'} j + \frac{u'}{K'} \partial_x^2 j = \frac{i\omega}{\pi} E_{ext} \tag{6.31}$$

To find the intrinsic parameters of our wire, we map our intrinsic equation (Eq. 6.25) to this standard form (Eq. 6.31). We identify the effective parameters u', K' by matching coefficients:

1. Matching the $\partial_x^2 j$ coefficient:

$$\frac{u'}{K'} = v_F \tag{6.32}$$

2. Matching the j coefficient (the inertial term):

$$\frac{1}{u'K'} = \frac{1}{uK} \implies u'K' = uK \tag{6.33}$$

We now solve this system for the effective interaction parameter K'. Multiply Eq. 6.32 and Eq. 6.33:

$$\left(\frac{u'}{K'}\right)(u'K') = v_F(uK) \implies (u')^2 = uKv_F \implies u' = \sqrt{uKv_F} \tag{6.34}$$

Divide Eq. 6.33 by Eq. 6.32:

$$\frac{u'K'}{u'/K'} = \frac{uK}{v_F} \implies (K')^2 = \frac{uK}{v_F} \tag{6.35}$$

Taking the square root gives the effective Luttinger parameter for the intrinsic response:

$$K' = \sqrt{\frac{uK}{v_F}} \tag{6.36}$$

This explicitly proves Eq. (16) of Safi [5].

6.4.2 Final Result

The DC conductance g' is given by the standard formula using the effective parameter K':

$$g' = K'\frac{2e^2}{h} = \sqrt{\frac{uK}{v_F}}\frac{2e^2}{h} \tag{6.37}$$

For a Galilean invariant system where $uK = v_F$, this reduces to $g' = 2e^2/h$. However, in the general case (e.g., lattice models) where $uK \neq v_F$, the intrinsic conductivity is renormalized.

6.5 Conclusion

We have rigorously proven that the intrinsic response of an isolated quantum wire to a local field leads to a renormalized conductance $g' = K'(2e^2/h)$. This contrasts with the Landauer-Büttiker result $g = 2e^2/h$ derived in Chapter 5. The difference arises purely from the definition of the measurement: Chapter 5 utilized boundary conditions imposed by Fermi-liquid leads which "wash out" the interaction parameters at the contacts, whereas this chapter analyzed the self-consistent response of the bulk fluid to internal fields. The two formalisms are consistent when applied to their respective physical domains.

1. **Measurement Setup Matters:** Transport measurements always involve leads. In a realistic experiment (Case 1), the leads are Fermi liquids ($K = 1$). The interactions inside the wire are "washed out" at the boundaries, enforcing $T = 1$ and $g = 2e^2/h$.

2. **The Kubo Calculation:** The standard "renormalized" Kubo result ($g = K2e^2/h$) is correct for an *infinite* Luttinger liquid responding to a field, or a wire connected to Luttinger liquid leads. It describes the intrinsic response of the interacting fluid.

3. **Consistency:** When the Kubo formalism is applied to the composite system (Wire + Fermi Liquid Leads), it correctly reproduces the Landauer result $g = 2e^2/h$.

Thus, there is no contradiction. The quantized conductance $2e^2/h$ observed in experiments is robust against electron-electron interactions in the wire, provided the contacts are good Fermi liquids. Deviations from this value (as seen in Yacoby's experiment) must stem from other sources, such as backscattering at the contacts due to impurities or poor matching, not from the intrinsic Luttinger liquid nature of the wire itself.

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