Landauer-Büttiker vs. Kubo Formalisms in Nanoscale Transport

Term Paper Presentation

Condensed Matter Physics II (PH5103)
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References


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The Central Question

What is the true conductance of a 1D interacting wire?

The Brocks and Nikolić papers demonstrate that for *non-interacting* electrons, the Landauer-Büttiker (scattering) and Kubo (linear response) formalisms are equivalent when applied correctly to an open system with leads.

The Problem

What happens when the conductor is a **1D interacting wire**?

- Do Landauer and Kubo still agree?
- How do strong electron-electron interactions (which are non-perturbative in 1D) affect the quantized conductance $g = 2e^2/h$?

The Physical System

We analyze transport through a 1D quantum wire, which acts as the sample or scattering region, connected to two macroscopic leads that are in turn connected to reservoirs.

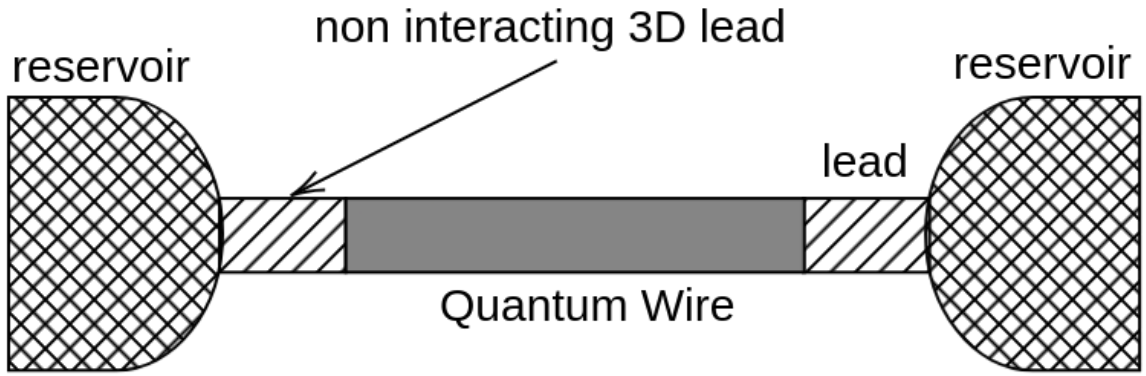

Figure: 3d leads connected to 1D interacting wire

- Becomes important for nanoscale transport (nearly perfect 1D wires like Carbon Nanotubes or Semiconductor Nanowires) where conductance is quantized, set by the number of open channels at E_f.
- Ideal thin wire with some finite thickness can have infinite conductance!

Two Scenarios $\Longrightarrow$ Two Answers

Case 1: Realistic (Wire + Leads)

- **System:** The interacting 1D wire is connected to non-interacting 3D leads.
- **Question:** What is the two-terminal conductance $g = I/V$?
- **Method:** Landauer-Büttiker formalism, which is defined by transmission T between reservoirs.
- **Answer:** $g = \frac{2e^2}{h}$

Case 2: Theoretical (Isolated Wire)

- **System:** An infinite, isolated 1D interacting wire.
- **Question:** What is the "intrinsic" conductivity σ of the material?
- **Method:** Kubo formalism, defined as a response to a local electric field E_{loc}.
- **Answer:** $g' \neq \frac{2e^2}{h}$

The core conflict: Why do these two apparently valid approaches give different answers? Safi's paper shows that the boundary conditions imposed by the leads are the deciding factor.

Formalisms at a Glance

Formula	Formalism	Valid (Isolated)?	Valid (Leads)?
$g = \frac{2e^2}{h} \sum_n T_n$	**Landauer-Büttiker** Conductance is *transmission*	No Needs reservoirs	**Yes** Designed for this
$g' = \frac{2e^2}{h} K'$	**Kubo (Intrinsic)** Conductance is *local response*	**Yes** Gives $g' \neq g$	No Ignores boundaries
$g = \frac{2e^2}{h} T$	**Kubo (Open System)** Full response of wire+leads	No Needs reservoirs	**Yes** Proven Computationally equivalent to via DMRG.

Table: Comparison of Landauer and Kubo formalisms for the two physical scenarios, which are not equal only for non-interacting case.

Note: 2 years prior: I. Safi and H. J. Schulz, Phys. Rev. B 52, R17040 (1995) focused on AC conductivity and showing that at finite frequencies, quantum wire could act like a Fabry-Pérot resonator.

Experimental Reality: Yacoby et al. (1996)

Safi's theory predicts $g = 2e^2/h$ for a clean wire connected to leads. However, experiments on ultra-clean quantum wires show a persistent deviation.

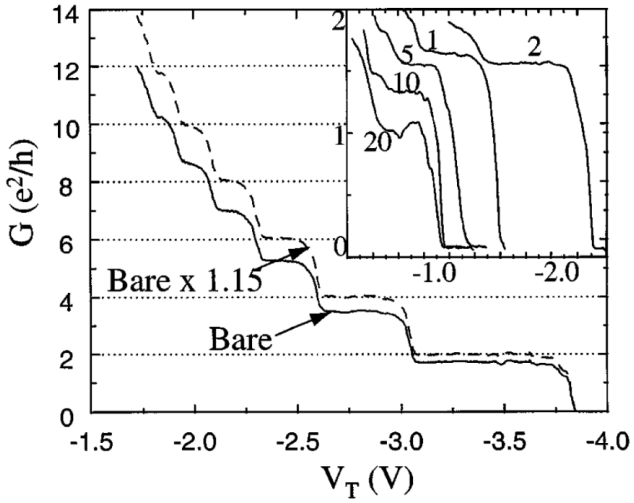

Figure: Conductance vs. Gate Voltage. Conductance is quantized but reduced by $\sim 25\%$ ($0.75 \times 2e^2/h$). Plateaus are flat (independent of density), ruling out simple disorder models.

The Conflict with Safi/Landauer:

- Yacoby argues the leads themselves (edge states) might be interacting (Non-Fermi Liquid).
- **Key Insight:** The coupling between the 2D reservoir and 1D wire is not "perfect transmission." Backscattering at the contact (due to momentum mismatch) likely causes the reduction.
- Recent experiments by Rech et al. (PRL, 2009) prove: $G = \frac{2e^2}{h}\left[1 - \frac{\pi^2}{12}\left(\frac{T}{\mu}\right)^2\right]$.

Safi's Response: The "Washing Out" effect relies on perfect, non-interacting leads. Real experiments have complex contacts that violate these assumptions.

Derivation of Landauer Formula (1D): The Setup

Consider a 1D wire of length L connected to two reservoirs. We model the electrons as traveling waves.

1. Density of States

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}; \qquad \psi_n = \frac{1}{i\sqrt{2L}}(e^{ik_n x} + e^{-ik_n x})$$

The wave functions are quantized in the wire: $k_n = n\pi/L$. The density of states (spacing between k points) is: $\Delta k = \frac{\pi}{L}$ For a single normalized traveling wave, the electron density is $\rho = |A|^2 = 1/2L$.

2. Current per State

An electron in state k moving with velocity v_k carries a current:

$$I_{state} = -e \cdot v_k \cdot \rho = -\frac{ev_k}{2L}$$

The net current arises only from the **bias window** created by the voltage U, shifting the chemical potentials such that $\mu_L - \mu_R = -eU$.

Derivation of Landauer Formula (1D): The Integral

We sum the current contributions from all states n within the energy bias window (E_F to $E_F - \Delta V$).

Total Current (I_{in}):

$$I_{in} = \underbrace{2}_{\text{spin}} \sum_k I_{state} = 2 \sum_k \left(-\frac{e v_k}{2L} \right)$$

Since L is macroscopic, we convert the discrete sum to an integral using the density of states ($\sum \to \frac{L}{\pi} \int dk$):

$$I_{in} = -\frac{e}{L} \left(\frac{L}{\pi} \int v(k) dk \right) = -\frac{e}{\pi} \int v(k) dk$$

Velocity Cancellation

We relate the particle velocity to the **group velocity** of the wave packet (Brocks, Eq 1.36).

Group Velocity Relation

$$v(k) = \frac{\hbar k}{m} = \frac{1}{\hbar}\frac{d(\frac{\hbar^2 k^2}{2m})}{dk} = \frac{1}{\hbar}\frac{dE}{dk}$$

Substituting this into our current integral allows us to change the integration variable from momentum (k) to Energy (E):

$$\int v(k)dk = \int \left(\frac{1}{\hbar}\frac{dE}{dk}\right) dk = \frac{1}{\hbar}\int dE$$

The dk terms cancel perfectly. This exact cancellation is unique to 1D transport, linking current directly to the energy range.

Derivation of Landauer Formula (1D): The Result

We integrate dE over the bias window $\Delta V = -eU$ (Brocks, Eq 1.37):

$$I_{in} = -\frac{e}{\pi}\left(\frac{1}{\hbar}\int_{E_F-\Delta V}^{E_F} dE\right) = -\frac{e}{\pi\hbar}(\Delta V) = \frac{e^2}{\pi\hbar}U$$

The transmitted current is the incident current times the transmission probability T:

$$I_{total} = I_{in}T = \frac{e^2}{\pi\hbar}UT$$

The Landauer Formula

Using $h = 2\pi\hbar$, the conductance $g = I/U$ becomes:

$$g = \frac{e^2}{\pi(h/2\pi)}T \implies \boxed{g = \frac{2e^2}{h}T}$$

Extension to 3D: The Origin of Channels

In a realistic nanowire, the electrons are confined in the transverse directions (y, z) but free to move along the wire (x).

1. Separation of Variables

The 3D Schrödinger equation separates because the transverse confinement $V(y, z)$ is independent of x (Brocks, Eq 2.2):

$$\Psi(x, y, z) = \phi(x)\chi_n(y, z)$$

$$E_{total} = E_x + E_n$$

2. "Particle in a Box" Quantization

The transverse wavefunction $\chi_n(y, z)$ satisfies a 2D particle-in-a-box equation:

$$\left[-\frac{\hbar^2}{2m}(\partial_y^2 + \partial_z^2) + V(y, z)\right]\chi_n = E_n\chi_n$$

This generates a discrete set of energy levels E_n (subbands). Each solution n is a **"transverse mode"** or **"channel"** since $\langle\chi_n|\chi_m\rangle = \delta_{mn}$ (no mixing).

Extension to 3D: Counting Open Channels

For an electron to transport current, it must have kinetic energy for motion in the x-direction ($E_x > 0$).

Condition for an "Open" Channel

Since $E_{total} = E_x + E_n$, the longitudinal energy is:

$$E_x = E_{total} - E_n$$

For a wave to propagate (k_x is real), we must have $E_{total} > E_n$. At the Fermi level ($E_{total} = E_F$), a channel n is **open** only if $E_F > E_n$.

Total Conductance: The total current is the sum of currents carried by all independent open modes (Brocks, Eq 2.9 - 2.11). Since each mode acts as a 1D wire with transmission T_n:

$$G = \frac{2e^2}{h} \sum_n T_n$$

Where the sum runs over the M modes where $E_n < E_F$. Modes with $E_n > E_F$ are "evanescent" (decaying) and do not conduct. This confirms that even in 3D, transport is quantized if the channels are ballistic ($T_n = 1$).

Why 1D is Special: Interacting Systems

2D/3D: Fermi Liquid

- In 2D or 3D, electrons can move *around* each other.
- The Coulomb repulsion is **screened**.
- The system behaves as a **Fermi Liquid**: electrons are "dressed" by interactions but still behave like independent quasiparticles.
- The single-particle Hamiltonian (like in the Brocks notes) is a valid approximation.

1D: Tomonaga-Luttinger Liquid

- In 1D, electrons **cannot** pass each other.
- Any push (interaction) creates a collective, many-body "sound wave" (a plasmon).
- The system is **strongly correlated**. The single-electron picture fails completely.
- We *must* use a many-body Hamiltonian to describe these collective modes.

Why 1D is Special

1D Excitations

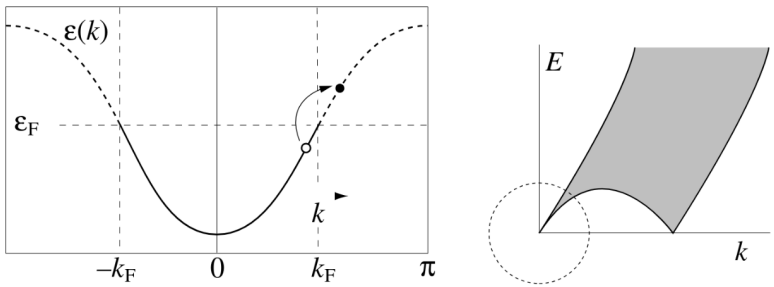

Figure: particle-hole excitations in one dimension(Sénéchal, Fig 3)

- A particle-hole pair with momentum k has a *sharp, well-defined* energy $\omega \approx v_F k$.
- They behave like coherent, stable particles (bosons).

2D Excitations

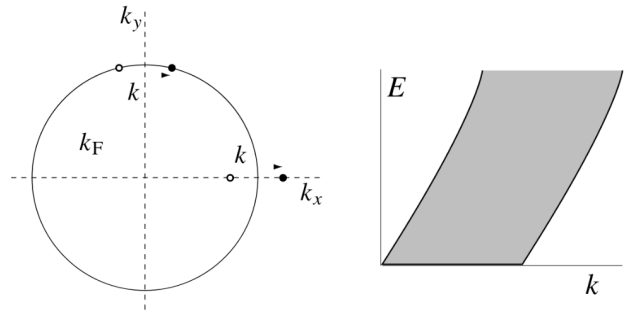

Figure: particle-hole excitations in two dimension(Sénéchal, Fig 4)

- A pair with momentum k has a *continuum* of possible energies.
- The excitation is not stable; it "decays."

The particle-hole excitations in 1D are unique, where the entire low-energy physics is described by these stable bosonic excitations (density waves). Bosonization works for the same reason.

The Washing Out Effect

Safi's key finding is that in a realistic measurement (Case 1), the complex interactions *inside* the wire are "washed out."

- The non-interacting leads "force" their properties onto the system at the boundaries.
- They act as perfect absorbers/injectors, forcing the transmission $T = 1$ for a perfect wire.
- The conductance $g = (2e^2/h)T$ is therefore determined *only* by the non-interacting leads, not by the internal (and complex) interactions of the wire itself.

The Role of Leads and Reservoirs

Why are the leads "non-interacting"?

- The leads are not 1D wires; they are the macroscopic 3D metal contacts.
- As in the previous slide, in 3D, electron-electron interactions are **screened** and can be neglected.
- The leads behave as a simple **Fermi Liquid** (a non-interacting electron gas).

The Non-Interacting Limit: $u = v_F$ and $K = 1$

- The physics of the 1D interacting liquid is described by two parameters:
  - u: The velocity of the "sound" (charge) waves.
  - K: The Luttinger parameter, measuring interaction strength.
- By definition, a **non-interacting** system is the limit where:
  - The interaction strength $K \to 1$.
  - The charge velocity $u \to v_F$ (the single-electron Fermi velocity).
- Thus, modeling the leads as "non-interacting" provides a simple, critical boundary condition: $K = 1, u = v_F$ in the leads.

Starting Point: The Fermionic Hamiltonian (H_0)

We start with the full many-body Hamiltonian in second quantization.

Kinetic (Non-Interacting) Term H_0

- All low-energy physics occurs near the Fermi points, $+k_F$ and $-k_F$.
- We linearize the dispersion: $E(k) \approx v_F(k - k_F)$ for right-movers and $E(k) \approx -v_F(k + k_F)$ for left-movers.
- We define two fermionic field operators:
  - $\psi_R(x)$: Annihilates a right-moving fermion.
  - $\psi_L(x)$: Annihilates a left-moving fermion.
- The kinetic Hamiltonian is (Sénéchal, Eq. 19):

$$H_0 = \int dx \left[-i\hbar v_F \psi_R^\dagger(x) \partial_x \psi_R(x) + i\hbar v_F \psi_L^\dagger(x) \partial_x \psi_L(x) \right]$$

The Fermionic Hamiltonian (H_{int})

The complexity comes from the interaction term H_{int}. In 1D, there are four key scattering processes ("g-ology").

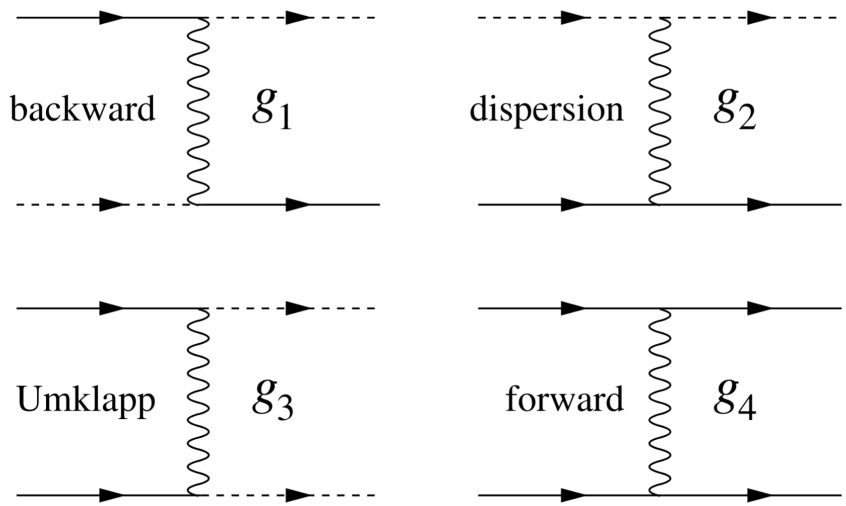

Figure: "g-ology" scattering (Sénéchal, Fig 2) with right-moving (continuous lines) and left-moving (dashed lines) electrons in one dimension.

- g_2 **(forward):** R-L scattering, they "pass through" each other.
- g_4 **(forward):** R-R or L-L scattering, they "disperse."
- g_1 **(backward):** R-L scattering, they "bounce back."
- g_3 **(Umklapp):** Two L-movers become two R-movers (or vice versa). Only relevant at half-filling.

The Problem (Fermionic Form)

The Tomonaga-Luttinger (TL) model simplifies this by only considering the forward scattering terms (g_2, g_4), which are the most dominant.

Interaction Densities

We define the density of right- and left-movers:

- $\rho_R(x) = \psi_R^\dagger(x)\psi_R(x)$
- $\rho_L(x) = \psi_L^\dagger(x)\psi_L(x)$

The Interacting TL Hamiltonian (Fermionic Form)

H_{int} is written in terms of these densities (Sénéchal, Eq. 27):

$$H_{int} = \int dx \left[g_2 \rho_R(x)\rho_L(x) + \frac{g_4}{2}(\rho_R(x)^2 + \rho_L(x)^2) \right]$$

The Problem

This is an Impasse!

The total Hamiltonian $H = H_0 + H_{int}$ is a **quartic** (four-fermion) interacting Hamiltonian.

- Standard methods from the Brocks notes (single-particle picture) fail.
- Perturbation theory fails in 1D because interactions are never "weak."

We cannot solve this Hamiltonian in its fermionic form.

Bosonization

Analogy: A Guitar String

- **Fermionic Picture (Hard):** Tracking all 10^{23} individual atoms (fermions). Impossible.
- **Bosonic Picture (Easy):** Describing the string by its collective vibrational modes.

Bosonization

Bosonization is mathematical mapping that rewrites the complicated, interacting 1D *fermion* system as a simple, *non-interacting* system of its collective density waves (bosons).

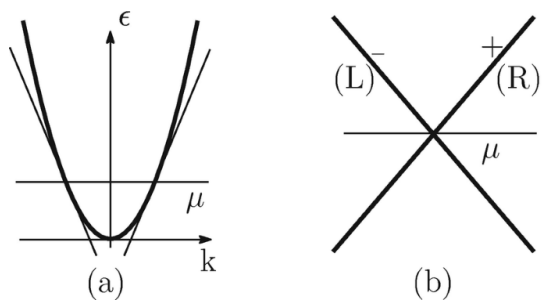

Figure: 1D fermions linearized dispersion with same result as Bosons following $E = u \cdot k$

Step 0: Bosonic Fields

We introduce two new bosonic fields, $\phi(x)$ and $\theta(x)$, which are "canonically conjugate".

Bosonic Field Definitions

The fermionic density and current can be exactly represented by these new fields:

- **Charge Density Field $\phi(x)$:**

$$\rho(x) = \rho_R + \rho_L = -\frac{1}{\sqrt{\pi}}\partial_x \phi(x)$$

  (A "bump" in density is a gradient in ϕ).

- **Phase (Current) Field $\theta(x)$:**

$$j(x) = v_F(\rho_R - \rho_L) = \frac{v_F}{\sqrt{\pi}}\partial_x \theta(x)$$

  (A current is a gradient in θ).

Both obey bosonic commutation relation: $[\phi(x), \partial_y \theta(y)] = i\pi\delta(x-y)$.

Step 0: Chiral Fields

To derive the Hamiltonian, we must map the collective fields (ϕ, θ) to the chiral fields (ϕ_R, ϕ_L) appearing in the fermion operators.

1. Chiral Decomposition (Sénéchal, Eq. 38)

We decompose the fields into Right (R) and Left (L) moving components:

$$\phi_R = \frac{1}{2}(\phi - \theta) \quad ; \quad \phi_L = \frac{1}{2}(\phi + \theta)$$

2. The Bosonization Identity (Sénéchal, Eq. 73)

The fundamental mapping relates the single fermion operator $\psi(x)$ to the exponential of the chiral boson field:

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a}} : e^{-i\sqrt{4\pi}\phi_R(x)} :$$
$$\psi_L(x) = \frac{1}{\sqrt{2\pi a}} : e^{+i\sqrt{4\pi}\phi_L(x)} :$$

(Where a is a lattice cutoff and :: denotes normal ordering).

Step 1: The Kinetic Hamiltonian (The Problem)

We wish to transform the free fermionic kinetic energy density:

$$\mathcal{H}_{0,R} = -i\hbar v_F \psi_R^\dagger(x)\partial_x \psi_R(x)$$

The Mathematical Challenge

We cannot simply plug the exponential definition into this equation.

1. $\psi_R^\dagger(x)$ and $\psi_R(x)$ are quantum operators.
2. The product of two field operators at the exact same point x is **singular** (diverges to ∞).

Point Splitting

We must evaluate the product at a small separation ϵ, and carefully take the limit $\epsilon \to 0$:

$$\psi_R^\dagger(x)\partial_x \psi_R(x) = \lim_{\epsilon \to 0} \left[\psi_R^\dagger(x+\epsilon)\partial_x \psi_R(x)\right]$$

Step 2: Point Splitting & OPE

We calculate the product using the **Baker-Campbell-Hausdorff (BCH)** formula for exponentials: $e^A e^B =: e^{A+B} : e^{\langle AB \rangle}$.

Operator Product Expansion (OPE) (Sénéchal, Eq. 83)

$$\psi_R^\dagger(x+\epsilon)\psi_R(x) \propto \underbrace{e^{\langle \cdots \rangle}}_{\text{Correlation}} \times \underbrace{: e^{i\sqrt{4\pi}(\phi_R(x+\epsilon)-\phi_R(x))} :}_{\text{Operator}}$$

Expanding for small ϵ:

$$\psi_R^\dagger(x+\epsilon)\psi_R(x) \approx \underbrace{\frac{1}{2\pi i \epsilon}}_{\text{Singularity}} + \underbrace{\frac{1}{\sqrt{\pi}}\partial_x \phi_R}_{\text{Current}} + \underbrace{i\epsilon : (\partial_x \phi_R)^2 :}_{\text{Energy}}$$

When we apply the derivative $\partial_x \psi \approx \frac{\psi(x+\epsilon)-\psi}{\epsilon}$ and subtract the vacuum energy (the singularity), only the quadratic term survives:

$$-i\psi_R^\dagger \partial_x \psi_R = \pi : (\partial_x \phi_R)^2 :$$

Step 3: The Free Boson Hamiltonian

Using the result from the OPE, the total kinetic Hamiltonian is:

$$H_0 = \hbar v_F \pi \int dx \left[: (\partial_x \phi_R)^2 : + : (\partial_x \phi_L)^2 : \right]$$

We now convert back to the physical ϕ, θ fields using $\phi_{R/L} = \frac{1}{2}(\phi \mp \theta)$:

$$(\partial_x \phi_R)^2 + (\partial_x \phi_L)^2 = \frac{1}{4}(\partial_x \phi - \partial_x \theta)^2 + \frac{1}{4}(\partial_x \phi + \partial_x \theta)^2$$

The cross terms $-2\partial\phi\partial\theta$ and $+2\partial\phi\partial\theta$ cancel out:

$$= \frac{1}{2}[(\partial_x \phi)^2 + (\partial_x \theta)^2]$$

Result: The Free Hamiltonian

$$H_0 = \frac{\hbar v_F}{2} \int dx \left[(\partial_x \phi)^2 + (\partial_x \theta)^2 \right]$$

Step 4: Adding Interactions

In the Tomonaga-Luttinger model, electrons interact via density-density repulsion ($H_{int} = \int dx[g_2\rho_R\rho_L + g_4/2(\rho_R(x)^2 + \rho_L(x)^2]$).

- In boson language: $\rho_R\rho_L \propto (\partial_x\phi - \partial_x\theta)(\partial_x\phi + \partial_x\theta) = (\partial_x\phi)^2 - (\partial_x\theta)^2$.
- $H_{int} = \int dx \left[C_1(\partial_x\phi)^2 + C_2(\partial_x\theta)^2\right]$.
- Effect of interactions g_2 and g_4 reduced to effectively renormalize boson Hamiltonian coefficients.

We define two new parameters: **u** (velocity) and **K** (interaction).

The Final Hamiltonian (Sénéchal, Eq. 115)

$$H = \frac{\hbar}{2\pi} \int dx \left[\frac{u}{K}(\partial_x\phi)^2 + uK(\partial_x\theta)^2\right]$$

- **Repulsive Interactions:** Make density waves stiffer ($1/K \uparrow$) and current easier ($K \downarrow$). Thus, $K < 1$.
- **Non-Interacting Limit:** $u \to v_F$ and $K \to 1$.

Arriving at Safi's Equation of Motion

The Solved Hamiltonian

We regroup the new coefficients (coeff_A, coeff_B) into two physical parameters:

- u: The new, renormalized velocity of the charge waves.
- K: The Luttinger parameter, encoding the interaction strength.

The final Hamiltonian (Sénéchal, Eq. 115) is:

$$H = \frac{\hbar}{2\pi} \int dx \left[\frac{u}{K} (\partial_x \phi)^2 + uK (\partial_x \theta)^2 \right]$$

Or, in the (j, ρ) language of Safi (Eq. 4):

$$H_{TL} = \int \frac{\pi dx}{2uK} [j^2 + u^2 \rho^2]$$

Arriving at Safi's Equation of Motion

The total Hamiltonian H is the sum of the standard Tomonaga-Luttinger (TL) model (which accounts for kinetic energy and short-range interactions) and an explicit long-range interaction potential $U(x, y)$.

$$H = \underbrace{\int dx \frac{\pi}{2uK}[j^2 + u^2\rho^2]}_{H_{TL}} + \underbrace{\frac{1}{2}\iint dxdyU(x, y)\rho(x)\rho(y)}_{H_{\text{long-range}}}$$

Combine with $\partial_t A = \frac{1}{i}[A, H]$ to get:

Equation of Motion (Safi, Eq. 6)

$$\frac{\partial_t j(x, t)}{uK} + \partial_x \left\{ \frac{u}{K}\rho(x, t) + \int dyU(x, y)\rho(y, t) \right\} = 0$$

This is our new starting point for analyzing transport.

Finding Transmission

We solve the equation of motion for a stationary (DC) current.

- **1. Set $\partial_t = 0$ for DC Transport**
  The equation of motion (Safi, Eq. 6) becomes:

$$\frac{0}{uK} + \partial_x \left\{ \frac{u}{K}\rho(x) + \int dy U(x,y)\rho(y) \right\} = 0$$

- **2. Integrate over x**
  The derivative $\partial_x\{\dots\}$ is zero, which implies the quantity in the curly brackets must be a **constant** everywhere in space:

$$\frac{u}{K}\rho(x) + \int dy U(x,y)\rho(y) = C$$

Finding Transmission

- **3. Apply Boundary Conditions (The Leads)**
  This constant C can be found by evaluating the equation at a point x deep inside the non-interacting leads.

  At this point, we know:
  - $U(x, y) = 0$ (no interactions in the lead)
  - $u = v_F$ (non-interacting velocity)
  - $K = 1$ (non-interacting stiffness)

  Plugging these into our equation:

$$\frac{v_F}{1}\rho(x) + 0 = C \implies v_F \rho_{lead} = C$$

  This constant C is simply the injected current from the reservoir, j_{in}. Since $j_{in} = C$ in the leads, and C is constant everywhere, the current must be constant throughout the entire system (leads + wire), so:

$$T = 1 \qquad \text{(transmission)}$$

Solving for $T = 1$

Law 1: Safi's Equation

The quantity $C_1 = \frac{u}{K}\rho + \int U\rho$ must be constant.

- At Left Lead ($x \to -\infty$): $C_1 = v_F(\rho_+ + \rho_-)$
- At Right Lead ($x \to +\infty$): $C_1 = v_F(\rho_T)$

$$\implies v_F(\rho_+ + \rho_-) = v_F(\rho_T) \implies \rho_+ + \rho_- = \rho_{\mathbf{T}} \quad \text{(Eq. A)}$$

Law 2: Current Conservation (Continuity Eq.)

The net current $C_2 = j(x)$ must be constant.

- At Left Lead: $C_2 = v_F(\rho_+ - \rho_-)$
- At Right Lead: $C_2 = v_F(\rho_T)$

$$\implies v_F(\rho_+ - \rho_-) = v_F(\rho_T) \implies \rho_+ - \rho_- = \rho_{\mathbf{T}} \quad \text{(Eq. B)}$$

Solution

Subtracting (B) from (A) $\implies (\rho_+ + \rho_-) - (\rho_+ - \rho_-) = 0$

Conductance of a Perfect Interacting Wire

Theorem (Conductance of a Perfect Interacting Wire)

For a two-channel (spin $\uparrow, \downarrow$) wire, the Landauer formula (applied at the non-interacting leads, not inside the wire) is:

$$g = \frac{2e^2}{h} T$$

Since the leads force $T = 1$, the conductance is perfectly quantized:

$$g = \frac{2e^2}{h}$$

The "Washing Out" Effect

The internal interaction parameters (u, K) of the wire are washed out and do not appear in the final DC conductance.
Implication: If an experiment measures $g < 2e^2/h$, it is due to **impurities** or **imperfect contacts** (which cause $T < 1$), not the e-e interactions.

Case 2: Intrinsic Response (Summary)

What if we remove the leads and measure the response to a *local* field E_{loc}?

Standard Response (Leads)

Response to external field E_{ext}. Fixed boundary conditions.

$$\text{Coeffs} \to g = \frac{2e^2}{h}$$

Local Response (Intrinsic)

Response to local field E_{loc} (includes screening).

$$\text{Coeffs} \to g' = K' \frac{2e^2}{h}$$

Result:

$$K' = \sqrt{\frac{uK}{v_F}} \neq 1$$

Which can be derived by applying our main Bosonized Hamiltonian (Safi, Eq. 4) without the long range interacting term.
Intrinsic conductance *is* renormalized by interactions. The "Washing Out" is an artifact of the leads' boundary conditions.

(Full derivation in Appendix)

What We Have Covered

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5. **The Solution: Bosonization**
6. **Case 1 (Leads): Interacting System Landauer-Büttiker**
7. **Case 2 (Isolated): Kubo Renormalization (Summary)**
8. **Appendix (Case 2)**


I. Safi. (1997). Conductance of a quantum wire: Landauer's approach versus the Kubo formula. *Physical Review B*, 55(12), R7331.


B. K. Nikolić. (2002). Quantum transport through mesoscopic disordered interfaces, junctions, and multilayers. *Journal of Physics: Condensed Matter*, 14(34), 7871.


G. Brocks. (2005). *ELECTRON TRANSPORT AT THE NANOSCALE* (Lecture Notes).


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Kubo Intrinsic Renormalization

Here, we calculate the intrinsic conductance of an *isolated* wire, with no leads.

- We apply a *local* electric field E_{loc} (which includes the effect of other electrons) and find the current j.
- Safi (Eq. 14) shows the equation of motion for j in response to E_{loc} is:

$$\frac{\omega^2}{uK} j + v_F \partial_{xx} j = -i\omega E_{loc}$$

Comparison

This equation looks complicated. Let's compare it to a different, simpler model: a wire with only *short-range* interactions (parameters u', K') responding to a simple *external* field E.
The equation of motion for this simpler model is (Safi, Eq. 15):

$$\frac{\omega^2}{u'K'} j + \frac{u'}{K'} \partial_{xx} j = -i\omega E$$

Kubo Intrinsic Renormalization (1): The Setup

To find the intrinsic conductance of an isolated wire, we start with the fundamental description of the Luttinger Liquid.

1. The Hamiltonian (Safi, Eq. 4)

The Hamiltonian depends on current (j) and density (ρ) fluctuations:

$$H = \frac{\pi}{2} \int dx \left[\frac{1}{uK} j(x)^2 + \frac{u}{K} \rho(x)^2 \right]$$

2. Time Evolution (Heisenberg Equation)

We find the equations of motion using $\partial_t A = \frac{i}{\hbar}[H_{tot}, A]$. We use the fundamental commutation relation derived from bosonization:

$$[\rho(x), j(y)] = -\frac{ie^2 v_F}{\pi\hbar} \delta'(x-y) \propto -i\delta'(x-y)$$

Kubo Intrinsic Renormalization (2): Response to External Field

First, consider a reference model with parameters u', K' responding to an **external** field E_{ext} (where $H_{ext} = \int \rho V_{ext}$).

A. The Force Equation: The time derivative of current ($\partial_t j$) is driven by internal pressure and the external field.

$$\partial_t j = \frac{i}{\hbar}[H + H_{ext}, j] = \underbrace{-\frac{u'}{K'}\partial_x \rho}_{\text{Internal Pressure}} + \underbrace{v_F E_{ext}}_{\text{External Force}}$$

B. The Wave Equation (Eq. 15): Take the time derivative (∂_t) again and use the continuity equation ($\partial_t \rho = -\partial_x j$) to eliminate ρ. In Fourier space ($\partial_t \to -i\omega$):

$$\frac{\omega^2}{u'K'} j + \frac{u'}{K'}\partial_{xx} j = -i\omega E_{ext} \quad \text{(Safi, Eq. 15)}$$

Kubo Intrinsic Renormalization (3): Response to Local Field

Now consider the actual problem: the response to the **local** field E_{loc}, which includes internal screening ($V_{loc} = \delta H_{int}/\delta\rho$).

A. Decomposition of Force: We split the total Hamiltonian force into a Kinetic part (v_F) and an Interaction part (V_{loc}).

$$\text{Force} \propto -\partial_x \left(\frac{\delta H_{kin}}{\delta\rho} + \frac{\delta H_{int}}{\delta\rho} \right) = -\partial_x (v_F \rho + V_{loc})$$

B. The Wave Equation (Eq. 14): The driving force is now the *bare* Fermi pressure plus the local field. The spatial stiffness is set by v_F (kinetic energy only), while the temporal inertia ($1/uK$) comes from the LH side of the EOM:

$$\frac{\omega^2}{uK} j + v_F \partial_{xx} j = -i\omega E_{loc} \quad \text{(Safi, Eq. 14)}$$

Renormalized Parameter K'

Our wire behaves *identically* to the model (Eq. 15) if we just match the coefficients.

- **Match the $\partial_{xx} j$ term (spatial derivative):**

$$v_F = \frac{u'}{K'}$$

- **Match the $\omega^2 j$ term (time derivative):**

$$\frac{1}{uK} = \frac{1}{u'K'}$$

Renormalized Parameter K'

Solve the 2x2 System for K'

1. From the first equation: $u' = v_F K'$
2. Substitute this into the second equation:

$$\frac{1}{uK} = \frac{1}{(v_F K')K'} = \frac{1}{v_F (K')^2}$$

3. Rearrange to solve for K':

$$v_F (K')^2 = uK$$

$$\implies K' = \sqrt{\frac{uK}{v_F}}$$

This is Equation 16 from the Safi paper.

Comparing K′ to Kubo's Conductance for LL

- The intrinsic conductance of the simple model (with parameter K') is a standard result from Luttinger Liquid theory:

$$g' = \frac{2e^2}{h} K'$$

- Since our real wire is mathematically identical to this simple model, its intrinsic conductance must also be:

$$g' = \frac{2e^2}{h} \sqrt{\frac{uK}{v_F}}$$

(This is Equation 3 from the Safi paper).

Conclusion

When calculated this intrinsic way (isolated wire, response to E_{loc}), the conductance **is renormalized** (changed) by the interactions.
This is because u and K are functions of the interaction strength, so $K' \neq 1$.

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